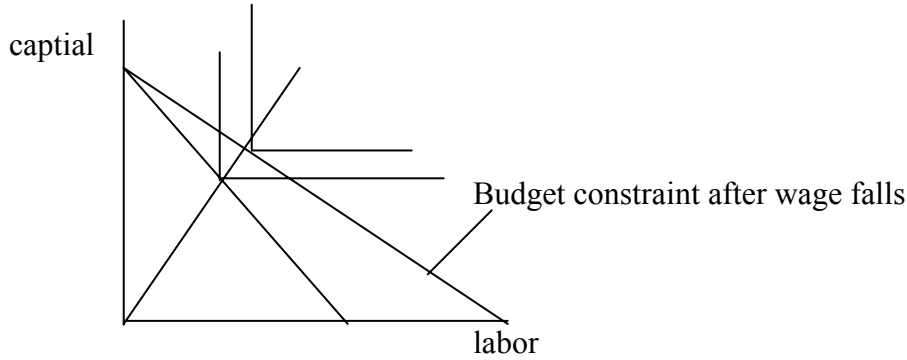
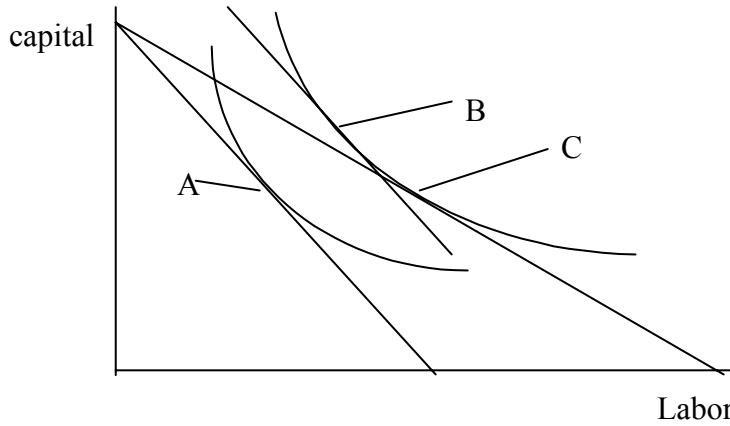


I. Scale and Substitution: The Diagram

Leontiff Case: Scale but no Substitution Effects as Wage Falls



General Case: Scale and substitution Effects as the Wage Falls



Movement from:

A to B \Rightarrow scale effects of the decrease in wage

B to C \Rightarrow substitution effect of the decrease in wage

II. Long-run demand for labor: factor substitution case and CRS: the general case

a. Introduction: scale and substitution effects

Assume CRS production function, so $Y=f(K,L)$ where we can multiply all the inputs by the same factor (for example, we could double the inputs) and the output would change by that same factor (continuing the example, output would double). But instead the inputs by 2, we will multiply it by $1/L$. Then by the fundamental property of CRS production functions we would have:

$$Y = f(L, K) \text{ CRS, implies } \frac{Y}{L} = f\left(\frac{L}{L}, \frac{K}{L}\right) \text{ or } Y = L f\left(1, \frac{K}{L}\right) = L F\left(\frac{K}{L}\right)$$

Rewrite this last expression, and take percentage changes to get

$$(*) \quad L = \frac{Y}{F\left(\frac{K}{L}\right)} \quad \text{so that} \quad d \ln L = d \ln Y - d \ln F\left(\frac{K}{L}\right)$$

This last expression (equation (*)) on the right above suggests what we are going to do: decompose the demand for L into a demand due to scale effects (through Y) and demand due to factor substitution (through $F(K/L)$). In order to make this discussion as unit-less as possible, all changes will be expressed as percentage changes (“d ln”-changes), including using the elasticity of product demand (in section b) and the elasticity of substitution (in section c).

b. Scale effects: Euler’s Theorem, Scale effects of input price changes

Section a indicates that for each one percentage change in output, there will be a one percentage change in the demand of labor. The demand for output depends on the price of the output, and other demand shifters, D:

$$Y = h(P, D) \quad \text{so} \quad d \ln Y = \frac{\partial \ln h}{\partial \ln P} d \ln P + \text{change due to demand shifter } D$$

“ $\frac{\partial \ln h}{\partial \ln P}$ ” is the elasticity of demand (for the output), which will be negative except for Giffen goods (which, as you recall, will be extraordinarily rare). We will denote the product elasticity of demand as η , i.e., $\frac{\partial \ln h}{\partial \ln P} \equiv \eta$.

To get the scale change in the demand of labor as wage and rental costs vary, we need to see how changes in these factor prices change the price of output. In CRS, average costs are equal to marginal costs are equal to the product price, so we have:

$$P = \frac{(sK + wL)}{Y} = s \frac{K}{Y} + w \frac{L}{Y}.$$

Totally differentiate this last expression, and rearrange terms to get

$$d \ln P = \frac{wL}{PY} d \ln w + \frac{sK}{PY} d \ln s + \left\{ \frac{wL}{PY} d \ln L + \frac{sK}{PY} d \ln K - \left(\frac{wL}{PY} + \frac{sK}{PY} \right) d \ln Y \right\}$$

Product price changes as input prices change (the first two right hand side terms) and as the use of labor and capital changes (the last three terms on the right hand side). Euler’s theorem implies that these three terms (the “{.}” stuff) exactly outsets each other, so that for CRS functions, these last three right hand side terms disappear and only input prices determine the output price. Since totally differentiating the production function yields

$$\frac{\partial \ln f}{\partial \ln L} d \ln L + \frac{\partial \ln f}{\partial \ln K} d \ln K = d \ln Y,$$

and

$$\frac{\partial \ln f}{\partial \ln L} = \frac{\partial f}{\partial L} \frac{L}{Y} = \frac{wL}{PY} \text{ if labor is paid the value of their marginal product,}$$

since $MP_L \cdot P = w$. A similar expression is available for capital. Hence, the only thing more that is needed is to show that labor's share and capital's share of the output sums to one. The Euler Theorem proves just that.

Euler's Theorem. We already discussed this theorem in the last part of lecture 3, where we showed in the one-input production function case that paying labor its marginal product exactly exhausts output in the CRS case. Euler's Theorem is an extension of that idea to the case where there are many inputs (we only do it in the case of two inputs here, but the result is general). Euler's Theorem for two inputs:

$$Y = \frac{\partial f}{\partial K} K + \frac{\partial f}{\partial L} L$$

some implications:

**implication 1: paying inputs their MP exactly exhausts the output produced:

Proof: just divide the expression above by the output, Y , to get

$$1 = \frac{\frac{\partial f}{\partial K} K}{Y} + \frac{\frac{\partial f}{\partial L} L}{Y} = v_K + v_L = \text{capital's share} + \text{labor's share}$$

So the share of output paid to capital plus the share of output paid to labor equals 1; real payments (in terms of physical product) to the factors of production exactly exhaust the output produced. For competitive factor input markets, it is the case that

$$v_L = \frac{wL}{PY} \text{ and } v_K = \frac{sK}{PY} \quad \text{Again, why?}$$

**implication 2: the change in product price can be written as:

$$d \ln P = \frac{wL}{PY} d \ln w + \frac{sK}{PY} d \ln s = v_L d \ln w + v_K d \ln s$$

Hence, the scale effect of a change in the input prices on the demand for labor is

$$(**) d \ln L = d \ln Y = \eta d \ln P = \eta (v_L d \ln w + v_K d \ln s) \quad (\text{scale effect alone})$$

c. Substitution effects

We have just finished with the $d \ln Y$ part of equation (*) above, we now turn to solving for the $d \ln F(\frac{K}{L})$ term of equation (*) at the first of this section. In order to discuss substitution effects, the most useful new concept will be that of the elasticity of substitution, which is the percentage change in the K/L ratio for a percentage change in the marginal rate of substitution. It is a measure of the rate at which capital is substituted for labor as the MRS changes. For the right-angle isoquant, the elasticity of substitution is zero. For the straight line isoquant cases (perfect substitutes), the elasticity of substitution is infinite. For continuous, curved, "normal" isoquants between these extremes, the elasticity of substitution is a positive real number. Define

$$\phi \equiv \frac{K}{L} \quad \text{and} \quad \rho \equiv MRS = \frac{MP_L}{MP_K}$$

so that the elasticity of substitution can be written as

$$\sigma = \frac{\frac{\partial \phi}{\phi}}{\frac{\partial \rho}{\rho}} = \frac{\partial \ln \phi}{\partial \ln \rho}$$

While this is defined for all production functions with two inputs, in non-homothetic production functions the elasticity of substitution will depend on the level of output, Y, as well as the K/L ratio: $\rho = \rho(\phi, Y)$. For homothetic functions, MRS depends only on the K/L ratio: $\rho = \rho(\phi)$, and not on the level of output. In equilibrium, costs are minimized where $\rho \equiv MRS = w/s$. So sometimes the elasticity of substitution is written as:

$$\sigma = \frac{\partial \ln \phi}{\partial \ln \rho} = \frac{\partial \ln \phi}{\partial \ln(w/s)} \quad \text{So that} \quad d \ln \phi = d \ln \frac{K}{L} = \sigma d \ln(w/s)$$

Since

$$-d \ln F(K/L) = -\frac{\partial \ln F(\phi)}{\partial \ln \phi} d \ln \phi$$

and we have an expression for $d \ln \phi$, what remains is to find out what $\frac{\partial \ln F(\phi)}{\partial \ln \phi}$ is.

We do this by differentiate Y with respect to K:

$$\frac{\partial Y}{\partial K} = \frac{\partial \{L F(\phi)\}}{\partial K} = L \frac{\partial F}{\partial \phi} \frac{1}{L} = \frac{\partial F}{\partial \phi} = MP_K$$

$$\text{Hence} \quad \frac{\partial \ln F(\phi)}{\partial \ln \phi} = \frac{\partial F}{\partial \phi} \frac{\phi}{F} = MP_K \frac{\phi}{F} = MP_K \frac{K/L}{Y/L} = MP_K \frac{K}{Y} = v_K.$$

$$\text{So} \quad -d \ln F(K/L) = -\frac{\partial \ln F(\phi)}{\partial \ln \phi} d \ln \phi = -v_K \sigma (d \ln w - d \ln s),$$

/and since shares sum to one, the substitution effect on the demand for labor equals

$$(***) \quad -(1 - v_L) \sigma (d \ln w - d \ln s) \quad (\text{substitution effect alone})$$

As the relative price of labor increases, the demand for labor falls. The fall is greater when σ is larger (so it is easier to substitute capital for labor) and when labor's share is greater (so the cost consequences of an increase in wages are greater). Putting the scale (***) and substitution effects (***) together, and collecting terms, we have

$$d \ln L = [\eta v_L - (1 - v_L) \sigma] d \ln w + [\eta v_K + v_K \sigma] d \ln s + \text{other demand shifters}$$

III. Hick's rules of derived demand

For the first three rules (Hick's rules number 1, 2 and 4) we define the elasticity of the demand for labor with respect to the wage as η_L (holding the price of capital constant), which from the last equation immediately above can be written as:

$$\eta_L = (v_L - 1) \sigma + v_L \eta$$

1. η_L is higher (more elastic) when the product price elasticity is higher (more elastic):

$$\frac{\partial \eta_L}{\partial \eta} = v_L > 0$$

2. η_L is higher (more elastic) when the elasticity of substitution is larger:

$$\frac{\partial \eta_L}{\partial \sigma} = (v_L - 1) < 0$$

4. η_L is higher (more elastic) the larger the cost of employing labor as a share of the total costs:

$$\frac{\partial \eta_L}{\partial v_L} = \sigma + \eta$$

Actually, this is true only if $-\eta > \sigma$ (so that the scale effect outweighs the substitution effect). If the substitution effect outweighs the scale effect, then it is false.

3. η_L is higher (more elastic) when the supply of the other factor is more elastic.

This is true, but is more difficult to show because we have to introduce supply functions into the analysis, and then treat market equilibrium to get the reduced form solution. So we went try to do this, since we are concerned here with derived demand.

IV. The Cobb Douglas Case: Deriving all this stuff for a specific production function

The Cobb Douglas Production Function is

$$Y = L^\alpha K^\beta \text{ where } \alpha + \beta = 1 \text{ (for CRS; } \alpha + \beta < 1 \text{ for DRS, etc.)}$$

Proof that it is CRS:

$$(2L)^\alpha (2K)^\beta = 2^{\alpha+\beta} L^\alpha K^\beta = 2 Y$$

“ β ” is capitals share of output ($v_K = MP_K K/Y$) and “ α ” is labor’s share ($v_L = MP_L L/Y$).

First order conditions:

$$MRS = \frac{MP_L}{MP_K} = \frac{w}{s} \text{ or } \frac{\alpha K^\beta L^{\alpha-1}}{\beta L^\alpha K^{\beta-1}} = \frac{\alpha K}{\beta L} = \frac{w}{s}$$

Notice that the MRS depends only on the K/L, and not on Y as is the case for non-homothetic functions. The last right hand side equality can be rewritten as

$$\beta wL = \alpha sK$$

This equation holds whenever the firm is optimally using its inputs (minimizing its cost). From these least two equations above, we can calculate the elasticity of substitution between capital and labor, and we can find the conditional input demand functions as well. The elasticity of substitution is $d \ln (K/L) / d \ln MRS$, but since is a function of the K/L ratio, we shall look at the inverse of the elasticity of substitution:

$$\frac{\partial \ln MRS}{\partial \ln (K/L)} = \frac{\partial \ln \left(\frac{\alpha K}{\beta L} \right)}{\partial \ln \left(\frac{K}{L} \right)} = 1. \text{ The inverse of the elasticity of substitution is one,}$$

so the elasticity of substitution for a Cobb-Douglas production function is one.

Substituting respectively L and K from “ $\beta wL = \alpha sK$ ” into the production function, we get

$$Y = \left(\frac{\alpha s}{\beta w} K \right)^\alpha K^\beta = \left(\frac{\alpha}{\beta} \right)^\alpha \left(\frac{s}{w} \right)^\alpha K$$

or, solving for K, we get

$$K = Y \left(\frac{\beta}{\alpha} \right)^\alpha \left(\frac{w}{s} \right)^\alpha$$

This is the conditional demand function for capital. A similar exercise (which you should do) provides the conditional demand function for labor as:

$$L = Y \left(\frac{\alpha}{\beta} \right)^\beta \left(\frac{s}{w} \right)^\beta,$$

which when we take logs and totally differentiate we get:

$$d \ln L = d \ln Y + \beta (d \ln s - d \ln w),$$

but since $d \ln Y = \eta (v_L d \ln w + v_K d \ln s)$ and $\beta = v_K$, we get (recall $\sigma = 1$ for this Cobb-Douglas technology)

$$d \ln L = \eta (v_L d \ln w + v_K d \ln s) + \sigma v_K (d \ln s - d \ln w)$$

just as we did in the general case. The demand for labor increases proportionately with an increase in output, falls as wage increases, and rises as the price of capital increases.

Substituting the conditional inputs (for K, L) into the cost constraint provides the cost function as

$$C = w [Y (\frac{\alpha}{\beta})^\beta (\frac{s}{w})^\beta] + s [Y (\frac{\beta}{\alpha})^\alpha (\frac{w}{s})^\alpha] = Y w^\alpha s^\beta [\frac{1}{\alpha^\alpha \beta^\beta}]$$

As with the case for the Leontiff-technology cost function, taking the derivative of the Cobb-Douglas cost function with respect to the input price yields the conditional input demand function. Taking the derivative of the cost function with respect to input price always yields the conditional input demand function, regardless of technology.