Brigham Young University
Economics 581 – Advanced Macroeconomics

Understanding Structural Vector Autoregressions (SVARs)

Introduction

The Structural VAR methodology is widely used in macroeconomics to present evidence on the importance of various types of shocks on key macroeconomic variables. In its simplest form the SVAR approach uses two variables and imposes assumptions aimed at distinguishing between permanent (supply-side) shocks and temporary (demand-driven) shocks.

One shortcoming of the approach is that we have no way of knowing how well the methodology identifies these two types of shocks in the dataset, because we don't have any clearly identified shocks outside of the SVAR itself. One strand of recent literature that attempts to address this involves building macroeconomic models and constructing artificial data where the driving shocks are known exactly. The evidence on how well SVARs identify in this context is mixed and inconclusive. See Chari, Kehoe & McGratten (2005) and Christiano, Eichenbaum & Vigfusson (2006) for two opposing viewpoints.

A Structural Econometric Model

Consider a structural model where K variables are driven by K different shocks as shown below.

$$Y_t = A_0 E_t + A_1 E_{t-1} + A_2 E_{t-2} + \ldots \tag{1}$$

Where Y_t & E_t are $K \times 1$ vectors and the A_i are $K \times K$ matrices.

The errors, E_t, are assumed to be serially independent with mean zero and a constant variance-covariance matrix, Ω which is $K \times K$.

Using L to denote the lag operator we can rewrite (1) as:

$$Y_t = A_0 L^0 E_t + A_1 L^1 E_t + A_2 L^2 E_t + \ldots \tag{2}$$

This can be further abbreviated to:

$$Y_t = A(L) E_t \tag{3}$$

where $A(L)$ is the vector polynomial in L from (2).

The polynomial in (2) & (3) gives us the impulse response functions (IRFs) of each of the K elements of Y to each of the K shocks in E. If we set the k^{th} element of E_1 to one standard deviation, set all other values of E_1 to zero, and set the values of the elements of all other E matrices to zero we will generate the IRFs for all K variables to shock number k.

An Reduced-Form Econometric Model

We can estimate a vector autoregression (VAR) on the time-series of *Y*s and express it using the following notation:

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + ... + B_P Y_{t-P} + U_t \quad (4)$$

Where Y_t & U_t are $K \times 1$ vectors and the B_p are $K \times K$ matrices.

The errors, U_t, are assumed to be serially independent with mean zero and a constant variance-covariance matrix, Σ which is $K \times K$.

Using L to denote the lag operator we can rewrite (1) as:

$$B_0 L^0 Y_t - B_1 L^1 Y_t - B_2 L^2 Y_t - ... - B_P L^P Y_t = U_t \quad (5)$$

where B_0 is a $K \times K$ identity matrix

This can be further abbreviated to:

$$B(L) Y_t = U_t \quad (6)$$

where $B(L)$ is the vector polynomial from (2).

This representation can be converted to a vector moving-average (VMA) representation.

$$Y_t = B^{-1}(L) U_t = C(L) U_t \quad (7)$$

We can derive the elements of $C(L)$ via iterative substitution.

$$Y_t = C_0 U_t + C_1 U_{t-1} + C_2 U_{t-2} + ... \quad (8)$$

But also

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + ... + B_P Y_{t-P} + U_t$$

Hence $C_0 = I$

Substituting a lagged version of (1)

$$Y_t = B_1 (B_1 Y_{t-2} + B_2 Y_{t-3} + ... + B_P Y_{t-P-1} + U_{t-1}) + B_2 Y_{t-2} + ... + B_P Y_{t-P} + U_t$$

$$Y_t = U_t + B_1 U_{t-1} + B_1 (B_1 Y_{t-2} + B_2 Y_{t-3} + ... + B_P Y_{t-P-1}) + B_2 Y_{t-2} + ... + B_P Y_{t-P}$$

Hence $C_1 = B_1$

$$Y_t = U_t + C_1 U_{t-1} + C_1 B_1 Y_{t-2} + C_1 B_2 Y_{t-3} + ... + C_1 B_P Y_{t-P-1} + B_2 Y_{t-2} + ... + B_P Y_{t-P}$$

$$Y_t = U_t + C_1 U_{t-1} + (B_1 B_1 + B_2) Y_{t-2} + (B_1 B_2 + B_3) Y_{t-3} + ... + (B_1 B_{P-1} + B_P) Y_{t-P} + B_1 B_P Y_{t-P-1}$$

Substituting another lagged version of (1)

$$Y_t = U_t + C_1 U_{t-1} + (C_1 + B_2)(B_1 Y_{t-3} + B_2 Y_{t-4} + ... + B_P Y_{t-P-2} + U_{t-2})$$
$$+ (C_1 B_2 + B_3) Y_{t-3} + ... + (C_1 B_{P-1} + B_P) Y_{t-P} + C_1 B_P Y_{t-P-1}$$

$$Y_t = U_t + C_1 U_{t-1} + (C_1 + B_2) U_{t-2} + (C_1 + B_2)(B_1 Y_{t-3} + B_2 Y_{t-4} + ... + B_P Y_{t-P-2})$$
$$+ (C_1 B_2 + B_3) Y_{t-3} + ... + (C_1 B_{P-1} + B_P) Y_{t-P} + C_1 B_P Y_{t-P-1}$$

Hence $C_2 = C_1 + B_2$

$$Y_t = U_t + C_1U_{t-1} + C_2U_{t-2} + C_2B_1Y_{t-3} + C_2B_2Y_{t-4} + ... + C_2B_PY_{t-P-2})$$
$$+ (C_1B_2 + B_3)Y_{t-3} + ... + (C_1B_{P-1} + B_P)Y_{t-P} + C_1B_PY_{t-P-1}$$
$$Y_t = U_t + C_1U_{t-1} + C_2U_t + (C_2B_1 + C_1B_2 + B_3)Y_{t-3} + (C_2B_2 + C_1B_3 + B_4)Y_{t-4} + ...$$
$$+ (C_2B_{P-2} + C_1B_{P-1} + B_P)Y_{t-P} + (C_2B_{P-1} + C_1B_P)Y_{t-P-1} + C_2B_PY_{t-P-2}$$

Substituting another lagged version of (1)

$$Y_t = U_t + C_1U_{t-1} + C_2U_{t-2} + (C_2B_1 + C_1B_2 + B_3)(B_1Y_{t-4} + B_2Y_{t-4} + ... + B_PY_{t-P-3} + U_{t-3})$$
$$+ (C_2B_2 + C_1B_3 + B_4)Y_{t-4} + ... + (C_2B_{P-2} + C_1B_{P-1} + B_P)Y_{t-P} + (C_2B_{P-1} + C_1B_P)Y_{t-P-1} + C_2B_PY_{t-P-2}$$

Hence $C_3 = C_2B_1 + C_1B_2 + B_3$

By similar iteration we get a general formula:

$$C_i = \sum_{j=1}^{\max\{P,P-i\}} C_{i-j}B_j \tag{9}$$

Hence, if we estimate the VAR and get estimates of the *B*'s we can relatively easily derive the *C*'s in the VMA representation.

<u>Mapping between Reduced-Form and Structural Models</u>

We would like to recover the *A*'s from the estimated *B*'s. However, our reduced-from shocks (*U*) are not the same as the structural shocks (*E*). So we need to impose some identifying restrictions.

First, since the size of the effects of our *E* shocks on our *Y*'s is determined entirely by the *A*'s, we can without loss of generality assume that the variances are one. Similarly, the correlation of the *Y*'s is also determined by the *A*'s and we can also assume the covariances of the elements in *E* are all zero. These two assumptions give us $\Omega = I$.

Comparing (1) and (9)

$$Y_t = A_0E_t + A_1E_{t-1} + A_2E_{t-2} + ... = U_t + C_1U_{t-1} + C_2U_{t-2} + ...$$

Gives $A_0E_t = U_t$ and $A_i = A_0C_i$

Hence, we need only to identify the A_0 matrix to get all the elements of *A*(*L*).

Since $A_0E_t = U_t$ we have $E\{A_0'E_t'E_tA_0\} = E\{U_t'U_t\}$ or $A_0'\Omega A_0 = \Sigma$, but $\Omega = I$, so we have:

$$A_0'A_0 = \Sigma \tag{10}$$

There are K^2 elements in A_0, but only $\frac{1}{2}K(K+1)$ unique elements in Σ, because it is a symmetric matrix. Hence we need to impose $\frac{1}{2}(K^2 - K)$ additional restrictions in order to identify A_0 from the estimated *B*'s and Σ.

Identification using Cholesky Assumptions

One way to come up with these additional restrictions is the Cholesky decomposition. In our context this boils down to an assumption that some of the K variables are contemporaneously affected by a particular shock, but others are affected only after a lag of one period.

To see this, consider the special case where K=2. We need $\frac{1}{2}(K^2 - K) = \frac{1}{2}(4-2) = 1$ additional restrictions. We could use the Cholesky decomposition to set either the (1,2) or (2,1) element of A_0 to zero. If we set the (1,2) element to zero we would be assuming that the 1st element of Y did not respond to the 2nd shock contemporaneously. However, since the (2,1) element is not necessarily zero we are still allowing the 2nd element of Y to respond contemporaneously to the 1st shock. If Y consisted of unemployment and output growth and the shock were a demand-side and supply-side shock, respectively, then this would be assuming that supply shocks have no contemporaneous effect on unemployment.

Identification using Long-Run Assumptions

Another common method of identification is to impose long-run restrictions on the effects of shocks. For example, one might assume that demand-side shocks have no long-run effects on key variables. This is the scheme used by Blanchard & Quah (1989). They used unemployment and output growth as elements of Y. Their long-run assumption was that the temporary (demand-side) shock had no long-run effect on the *level* of output. Since they were using output growth in their estimation this became a restriction on the cumulative effect of growth rate changes. Mathematically, we can write this restriction as:

$$\sum_{i=0}^{\infty} A_i(1,2) = 0 \tag{11}$$

References

Blanchard, O. J. and D. Quah (1989) "The Dynamic Effects of Aggregate Demand and Supply Disturbances," American Economic Review, vol. 79, no. 4, pp. 655-73.

Chari, V. V., P .J. Kehoe and E. R. McGrattan (2005) "A Critique of Structural VARs using Business Cycle Theory," Federal Reserve Bank of Minneapolis, Research Department Staff Report 364.

Christiano, L. J, M. Eichenbaum and R. Vigfusson (2006) "Assessing Structural VARs," mimeo.

Gali, J. (1989) **"**Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?" American Economic Review, vol. 89, no. 1, pp. 249-71.

Kilian, L. (1998) "Small-Sample Confidence Intervals for Impulse Response Functions," Review of Economics and Statistics, vol. 80, no. 2, pp. 218-30.