A Simple Real Business Cycle Model
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Major Features of the Model

Two sources of uncertainty: y & z
Stochastic growth about a deterministic trend
Simple labor-leisure decision
Deterministic population growth

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
y	shock to consumer confidence (temporary)
K	capital stock owned by household
J	capital demanded by firms
C	consumption
N	size of population
ℓ	labor supplied per worker
L	labor demanded by firms
H	fixed number of households
w	wage rate
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
g	trend in z
n	grow of population
σ	elasticity of substitution, σ>0
ρ	autocorrelation parameter for z; 0<ρ<1
ψ	autocorrelation parameter for y; 0<ψ<1

We next setup the timing of the model, where a prime on a variable indicates its value next period.

1) Beginning of period – y & z known
2) Factor markets open & clear
   K is loaned out to production firms and r is determined
   L is hired out to production firms and w is determined
3) Production of goods occurs
4) Factor payments made (wL, rK)
5) K' is chosen
   Consumption, C, occurs
6) Temporary shocks, y' & z' revealed

End of period

Nonstationary Model

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, y, z\}$, the household solves the following non-linear program when the factor markets clear. *C* is total consumption, summing up consumptions by all *N* individuals in the household. Consumption per member is *C*/*N*. For notational ease define $c \equiv C/N$. *w* is wages per worker.

$$V(K,\mathbf{\Omega}) = \underset{K',\ell}{Max}\, u(c,1-\ell)\tfrac{N}{H} + \beta E\{e^{y}V(K',\mathbf{\Omega}')\}$$

where:

$$C = w\ell N + (1-\delta+r)K - K' \tag{1.1}$$

or

$$c = \tfrac{C}{N} = w\ell + (1-\delta+r)\tfrac{K}{N} - \tfrac{K'}{N} \tag{1.1'}$$

The first-order conditions from the maximization problem are:
One condition for *K*':

$$u_C(c,1-\ell)(-\tfrac{1}{N})\tfrac{N}{H} + \beta E\{e^{y}V_K(K',\mathbf{\Omega}')\} = 0$$

One condition for ℓ

$$u_\ell(c,1-\ell)(-1)\tfrac{N}{H} + u_C(c,1-\ell)w\tfrac{N}{H} = 0$$

The envelope condition from this problem is as follows.
One condition for *K*:

$$V_K(K,\Omega) = u_C(c,1-\ell)(1-\delta+r)\tfrac{1}{N}\tfrac{N}{H}$$

The Euler equations are:
One condition for *K*':

$$u_C(c,1-\ell) = E\{e^{y}u_C(c',1-\ell')(1-\delta+r')\} \tag{1.2}$$

One condition for ℓ

$$u_\ell(c,1-\ell) = u_C(c,1-\ell)w \tag{1.3}$$

Picking functional form of $u(c) = \ln c + b\ln[e^{gt+z}(1-\ell)]$

$$u_C(c,1-\ell) = c^{-1}$$

$$u_\ell(c,1-\ell) = b[e^{gt+z}(1-\ell)]^{-1}e^{gt+z}(-1)$$

Rewriting (1.2) and (1.3) using the functional forms above

$$c^{-1}\tfrac{1}{N} = \beta E\{e^{y}(c')^{-1}(1-\delta+r')\}$$

$$1 = \beta E\{e^{y}(\tfrac{c}{c'})(1-\delta+r')\} \tag{1.2'}$$

$$b[e^{gt+z}(1-\ell)]^{-1}e^{gt+z} = c^{-1}w$$

$$bc = (1-\ell)w \tag{1.3'}$$

Exogenous Laws of Motion

The law of motion for z is:
$z' = \rho z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.4)

The law of motion for y is:
$y' = \psi y + \varepsilon_y'$; where ε_y' is distributed normal with a mean of 0 and a variance of σ_y^2 (1.5)

The law of motion for N is:
$$N' = (1+n)N = (1+n)^t N_0 \doteq e^{nt} N_0;\ N_0 = 1 \tag{1.6}$$

Final Goods Producers

Final goods production is:
$$Y = J^{\alpha}(e^{gt+z}L)^{1-\alpha} \tag{1.7}$$

Each period goods producers solve the following unconstrained maximization problem.
$$\underset{k',\ell}{Max}\ J^{\alpha}(e^{gt+z}L)^{1-\alpha} - rJ - wL$$
The first-order conditions from this problem reduce to:
$$wL = (1-\alpha)Y \tag{1.8}$$
$$rJ = \alpha Y \tag{1.9}$$

Market Clearing

There are three markets: goods, capital and labor. By Walras Law we need only examine two: capital and labor.

$$J = K \tag{1.10}$$
$$L = N\ell \tag{1.11}$$

Eqs (1.1), (1.2'), (1.3') & (1.4)-(1.11) are the system.

We can simplify the system somewhat by substituting (1.11) & (1.12) into (1.8) – (1.10) and eliminating variables *J* & *L*. Eliminate *c* by substituting *c*=*C*/*N*

$z' = \rho z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.4)

$y' = \psi y + \varepsilon_y'$; where ε_y' is distributed normal with a mean of 0 and a variance of σ_y^2 (1.5)

$$N' = (1+n)N \quad (1.6)$$

$$1 = \beta E\{e^y(\tfrac{C/N}{C'/N'})(1-\delta+r')\} \quad (1.2')$$

$$b\tfrac{C}{N} = (1-\ell)w \quad (1.3')$$

$$Y = K^{\alpha}(e^{gt+z}N\ell)^{1-\alpha} \quad (1.7)$$

$$wN\ell = (1-\alpha)Y \quad (1.8)$$

$$rK = \alpha Y \quad (1.9)$$

$$C = w\ell N + (1-\delta+r)K - K' \quad (1.1)$$

Equations (1.2'), (1.3'), (1.4), (1.5) & (1.6) are the dynamic equations. Equations (1.1) & (1.7) – (1.9) are useful definitions of non-state variables which could be substituted into the dynamic equations, but which are left as definitions for ease of notation. The state variables are z, y, N & K. z, y & N are exogenously generated outside the economic model. K is endogenously determined by consumer behavior. Note that ℓ is not a state variable since knowing z, y, N & K implicitly defines ℓ by (1.3') and the definitions. However, we could still treat it as a state variable if we wanted.

Since both *N* and the level of technology are growing, we have a non-stationary system. In order to linearize about a steady state, we need to transform this non-stationary system into a stationary one.

Transformation & Simplifications
We will eliminate N as a state variable by our transformation.

1) If z is stationary ($\rho < 1$):
Transform the problem by dividing all growing variables (except w) by $A \equiv e^{(g+n)t}$, denoting with a carat. w is divided by e^{gt}.

$z' = \rho z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (2.1)

$y' = \psi y + \varepsilon_y'$; where ε_y' is distributed normal with a mean of 0 and a variance of σ_y^2 (2.2)

$$1 = \beta E\left\{ e^{y} \frac{\hat{C}}{(1+g)\hat{C}'}(1-\delta+r') \right\} \tag{2.3}$$

$$(1-\ell)\hat{w} = b\hat{C} \tag{2.4}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}\ell)^{1-\alpha} \tag{2.5}$$

$$\hat{w}\ell = (1-\alpha)\hat{Y} \tag{2.6}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$

$$\hat{C} = \hat{w}\ell + (1-\delta+r)\hat{K} - (1+g+n)\hat{K}' \tag{2.8}$$

2) If z is non-stationary ($\rho = 1$):
Transform the problem by dividing all growing variables (except w) by $A \equiv e^{(g+n)t+z}$, denoting with a carat. w is divided by e^{gt+z}.

$\Delta z' = \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (2.1)

$y' = \psi y + \varepsilon_y'$; where ε_y' is distributed normal with a mean of 0 and a variance of σ_y^2 (2.2)

$$1 = \beta E\left\{ e^{y} \frac{\hat{C}}{(1+g+\Delta z')\hat{C}'}(1-\delta+r') \right\} \tag{2.3}$$

$$(1-\ell)\hat{w} = b\hat{C} \tag{2.4}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}\ell)^{1-\alpha} \tag{2.5}$$

$$\hat{w}\ell = (1-\alpha)\hat{Y} \tag{2.6}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$

$$\hat{C} = \hat{w}\ell + (1-\delta+r)\hat{K} - (1+g+n+\Delta z')\hat{K}' \tag{2.8}$$

Steady State

System of 8 equations in 7 unknowns, $\bar{z}$ or $\Delta\bar{z}, \bar{y}, \overline{C}, \overline{K}, \overline{w}, \overline{Y}, \bar{\ell}, \bar{r}$

Parameters are $\alpha, \beta, \delta, a, g, n, b$

Note that we could switch $\bar{r}$ and g.

$\bar{z} = 0$ or $\Delta\bar{z} = 0$ (3.1)

$\bar{y} = 0$ (3.2)

$$1 = \frac{\beta(1-\delta+\bar{r})}{1+g} \quad (3.3)$$

$$(1-\bar{\ell})\overline{w} = b\overline{C} \quad (3.4)$$

$$\overline{Y} = \overline{K}^{\alpha}\bar{\ell}^{1-\alpha} \quad (3.5)$$

$$\overline{w}\bar{\ell} = (1-\alpha)\overline{Y} \quad (3.6)$$

$$\bar{r}\overline{K} = \alpha\overline{Y} \quad (3.7)$$

$$\overline{C} = \overline{w}\bar{\ell} + (\bar{r}-\delta-g-n)\overline{K} \quad (3.8)$$

(3.1) – (3.2) eliminates $\bar{z}$ or $\Delta\bar{z}, \bar{y}$

(3.3) imposes restrictions on $\bar{r}, \beta, \delta, n, g$

Solve for β

$$\beta = \frac{1+g}{1-\delta+\bar{r}} \quad (3.5')$$

(3.5) into (3.6) & (3.7) eliminates $\overline{Y}$

$$\overline{w} = (1-\alpha)\overline{K}^{\alpha}\bar{\ell}^{-\alpha} \quad (3.6')$$

$$\bar{r} = \alpha\overline{K}^{\alpha-1}\bar{\ell}^{1-\alpha} \quad (3.7')$$

(3.6') into (3.4) eliminates $\bar{c}$:

$$\frac{(1-\bar{\ell})(1-\alpha)\overline{K}^{\alpha}\bar{\ell}^{-\alpha}}{b} = \bar{c} \quad (3.4')$$

(3.4') and (3.6') into (3.8)

$$\frac{(1-\bar{\ell})(1-\alpha)\overline{K}^{\alpha}\bar{\ell}^{-\alpha}}{b} = (1-\alpha)\overline{K}^{\alpha}\bar{\ell}^{1-\alpha} + (\bar{r}-\delta-g-n)\overline{K} \quad (3.8')$$

(3.7') and (3.8') are system in $\overline{K}$ and $\bar{\ell}$

Solving (3.7') for $\bar{\ell}$:

$$\bar{\ell} = \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\overline{K}$$

Substituting into (3.8')

$$\frac{\left[1-\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\overline{K}\right](1-\alpha)\overline{K}^{\alpha}\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\overline{K}\right]^{-\alpha}}{b}=(1-\alpha)\overline{K}^{\alpha}\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\overline{K}\right]^{1-\alpha}+(\bar{r}-\delta-g-n)\overline{K}$$

$$\left[1-\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\overline{K}\right](1-\alpha)\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{-\alpha}=b(1-\alpha)\overline{K}\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}+b(\bar{r}-\delta-g-n)\overline{K}$$

$$(1-\alpha)\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{-\alpha}-(1-\alpha)\overline{K}\left(\frac{\bar{r}}{\alpha}\right)=b(1-\alpha)\overline{K}\left(\frac{\bar{r}}{\alpha}\right)+b(\bar{r}-\delta-g-n)\overline{K}$$

$$(1-\alpha)\left[\left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{-\alpha}=\left\{(1+b)(1-\alpha)\left(\frac{\bar{r}}{\alpha}\right)+(\bar{r}-\delta-g-n)\right\}\overline{K}$$

$$\overline{K}=\frac{(1-\alpha)\left(\frac{\alpha}{\bar{r}}\right)^{\frac{\alpha}{1-\alpha}}}{(1+b)(1-\alpha)\left(\frac{\bar{r}}{\alpha}\right)+(\bar{r}-\delta-g-n)} \qquad (3.9)$$

Note we could also use the dynamic equations (2.1) – (2.8) to solve for the steady state numerically if we were too lazy to derive the closed form solutions, which we often are.

Dynamics

Exogenous state variables are $\bar{z}$ or $\Delta\bar{z}, y$

Endogenous state variables are $\hat{K}$

Exogenous shocks are $\varepsilon_y, \varepsilon_z$

All other variables are endogenous non-state variables. Uhlig calls these "jump"variables and sometimes they are only implicitly defined by the system. In our case ℓ is such a variable. We could treat it as a state variable, or we can use Uhlig's programs to solve for a linear approximation of how it depends on the exogenous and endogenous state variables.

Laws of motion:

$z' = \rho z + \varepsilon_z'$ or $\Delta z' = \varepsilon_z'$; $\varepsilon_z' \sim N(0, \sigma_z^2)$

$y' = \psi y + \varepsilon_y'$; $\varepsilon_y' \sim N(0, \sigma_y^2)$

Euler equations to linearize are:

$$1 = \beta E\left\{ e^y \frac{\hat{C}}{(1+g)\hat{C}'}(1-\delta+r') \right\} \tag{4.1a}$$

or

$$1 = \beta E\left\{ e^y \frac{\hat{C}}{(1+g+\Delta z')\hat{C}'}(1-\delta+r') \right\} \tag{4.1b}$$

$$(1-\ell)\hat{w} = b\hat{C} \tag{4.2}$$

The above are defined using the following as definitions:

$\hat{Y} = \hat{K}^{\alpha}(e^z \ell)^{1-\alpha}$ or $\hat{Y} = \hat{K}^{\alpha}\ell^{1-\alpha}$

$\hat{w} = \frac{(1-\alpha)\hat{Y}}{\ell}$

$r = \frac{\alpha \hat{Y}}{\hat{K}}$

$\hat{C} = \hat{w}\ell + (1-\delta+r)\hat{K} - (1+g+n)\hat{K}'$ or $\hat{C} = \hat{w}\ell + (1-\delta+r)\hat{K} - (1+g+n+\Delta z')\hat{K}'$

Proceed with the Method of Undetermined Coefficients

Using notation from Uhlig (1999), rather than Christiano (2002).

Exogenous state variables: z, y	vector Z_t
Endogenous state variables: K	vector X_t
Jump variables: ℓ	vector Y_t

note that since K was chosen last period and K' today, that K is part of X_{t-1} and K' is part of X_t

In matrix form:

Equation (4.2) is:

$$0 = AX_t + BX_{t-1} + CY_t + DZ_t$$

Euler Equation (4.1) is:

$$0 = E\{FX_{t+1} + GX_t + HX_{t-1} + JY_{t+1} + KY_t + LZ_{t+1} + MZ_t\}$$

Laws of motion are
$Z_{t+1} = NZ_t + \eta_{t+1}$

Where $X_t \equiv \left[\widetilde{K}_{t+1}\right]$ $Y_t = \left[\widetilde{\ell}_t\right]$ $Z_t \equiv \begin{bmatrix} z_t \\ y_t \end{bmatrix}$ $\eta_t \equiv \begin{bmatrix} \varepsilon_{zt} \\ \varepsilon_{yt} \end{bmatrix}$

We can obtain matrices *A* thru *M* using analytical or numerical derivatives.
Once we have them, we can use Uhlig's MATLAB package to get a linear approximation of the transition function in the form of:

$$X_{t+1} = PX_t + QZ_{t+1} \tag{4.3}$$

We can also get a linear approximation of the jump function

$$Y_{t+1} = RX_t + SZ_{t+1} \tag{4.4}$$

Parameterization

We need to choose parameters that are reasonable given evidence from micro studies or from other sources. Picking parameters so that the model matches empirical evidence is the simulation version of data mining.

α	.3 – average observed US capital share
δ	.02 – quarterly rate of depreciation
g	.008341 – average quarterly growth of GDP in post-war USA
n	.00125 – average annual growth of .5%
β	.98 – quarterly time discount factor
b	1.85 – chosen to give $\bar{\ell}$ equal to 40 hrs per 112 waking hours per week

Implies

$\bar{r}$	.0489 – implies a user cost of capital of 12.1% APR

Using equations (3.1) – (3.9) these give:

$\overline{K} = 8.4223$

$\overline{\ell} = 0.3438$

$\overline{C} = 0.6483$

$\overline{w} = 1.8275$

$\overline{Y} = .8975$

Parameterizing the laws of motion

ρ	.9
ψ	.9

The matrices in the transition function are:

$P = 0.9494$

$Q = \begin{bmatrix} -.8081 & 0.0528 \end{bmatrix}$

$R = -0.2199$

$S = \begin{bmatrix} -5.1923 & 0.2364 \end{bmatrix}$

Simulation

To simulate, use a random number generator to produce time-series for $\{\varepsilon_{zt}\}_{t=1}^{T}$ and $\{\varepsilon_{yt}\}_{t=1}^{T}$

Use these to generate the vector time-series $\{Z_t\}_{t=1}^{T}$

Then use (4.2) to generate the time-series $\{X_t\}_{t=1}^{T}$

Once we have $\{X_t\}_{t=1}^{T} = \{\widetilde{K}_t\}_{t=0}^{T-1}$ we can recover the stationary time-series for *K* using

$\hat{K}_t = \overline{K}(1+\widetilde{K}_t)$

This series, in turn, gives the non-stationary time-series using

$K_t = \hat{K}_t A_t$

with $A_t = e^{(g+n)t}$ in the stationary case

and $A_t = e^{(g+n)t+z_t}; z_t \equiv \Sigma_{s=1}^{t}\varepsilon_{zt}$ in the non-stationary case

Once we have these two series and the *y* time-series we can use the equations in the first section to find any other macroeconomic variable of interest.

Comparing Results

We generally compare our simulation to some set of data from a real economy. We should generate a large number of simulations and for each one generate an artificial data set with the same number of observations as the real data. We can then compare the standard deviations, correlations and autocorrelations from the model with those from the real data. A good model will match a large number of these statistical moments.

We need to filter the real data due to non-stationarity. We can use the exact filtering method on our simulated data which also has a growth component (chosen to match the real data). The alternative would be to use stationary values from the simulation, which are available. However, this raises the possibility that the empirical moments we report are sensitive to the filtering methodology. By using non-stationary data in both cases we introduce the exact same filtering distortion to both datasets.

Using variances of $\sigma_z^{\ 2} = .025$ and $\sigma_y^{\ 2} = 0$, HP filtering the moments of interest, and averaging over 1000 simulations gives the following:

	Y	C	I	K	L
st. dev.	0.0282	0.0368	0.0577	0.0054	0.0392
s.d. relative to Y	1	**1.3077**	**2.0434**	**0.1904**	**1.3872**
corr with Y(+4)	0.0502	-0.2109	0.4436	-0.3589	0.0727
corr with Y(+3)	0.2130	-0.1845	0.6911	-0.2081	0.2315
corr with Y(+2)	0.4234	-0.1408	0.9928	0.0172	0.4349
corr with Y(+1)	0.6872	0.3387	0.6544	0.3319	0.6882
corr with Y(0)	1	**0.8345**	**0.3753**	**0.5292**	**0.9987**
corr with Y(-1)	0.6872	0.6354	0.1575	0.6313	0.6707
corr with Y(-2)	0.4235	0.4560	-0.0064	0.6609	0.3973
corr with Y(-3)	0.2129	0.3033	-0.1229	0.6374	0.1818
corr with Y(-4)	0.0502	0.1778	-0.2017	0.5771	0.0176

References

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Uhlig, Harald. (1999), "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, Marimon and Scott, eds., Oxford University Press, pp. 30-61.