Ramsey-Cass-Koopmans Model in Discrete Time

Non-stationary forumlation

Household's objective function is:

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t \frac{1}{1-\theta} C_t^{1-\theta} \frac{L_t}{H} \tag{1.1}$$

Budget constraint is:

$$\sum_{t=0}^{\infty} \frac{1}{1+R_t} C_t \frac{L_t}{H} \leq \frac{K_0}{H}(1+r_0) + \sum_{t=0}^{\infty} \frac{1}{1+R_t} W_t \frac{L_t}{H} \; ; \; 1+R_t \equiv \prod_{s=0}^{t} 1+r_s \tag{1.2}$$

Technology is:

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} \tag{1.3}$$

Wages and interest rates are derivatives:

$$W_t = (1-\alpha) K_t^{\alpha} (A_t L_t)^{-\alpha} A_t \tag{1.4}$$

$$r_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} \tag{1.5}$$

Exogenous laws of motion for technology and labor supply are:

$$A_t = (1+g) A_{t-1} = (1+g)^t A_0 \tag{1.6}$$

$$L_t = (1+n) L_{t-1} = (1+n)^t L_0 \tag{1.7}$$

Capital accumulation equation with no depreciation is:

$$K_{t+1} = Y_t + K_t - C_t L_t \tag{1.8}$$

Transformed stationary formulation

Define stationary variables as follows:

$$c_t \equiv \frac{C_t}{A_t},\ k_t \equiv \frac{K_t}{A_t L_t};\ y_t \equiv \frac{Y_t}{A_t L_t},\ w_t \equiv \frac{W_t}{A_t}$$

The objective function and constraint become

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t \frac{1}{1-\theta} [c_t A_0 (1+g)^t]^{1-\theta} \frac{L_0 (1+n)^t}{H}$$

$$U = \frac{A_0 L_0}{H} \sum_{t=0}^{\infty} \left(\frac{(1+g)^{1-\theta}(1+n)}{1+\rho} \right)^t \frac{1}{1-\theta} c_t^{\ 1-\theta}$$

$$U = B \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\theta} c_t^{\ 1-\theta};\ B \equiv \frac{A_0 L_0}{H},\ \beta \equiv \frac{(1+g)^{1-\theta}(1+n)}{1+\rho} \tag{2.1}$$

$$\sum_{t=0}^{\infty} \frac{1}{1+R_t} c_t A_0 (1+g)^t \frac{L_0 (1+n)^t}{H} \le \frac{k_0 A_0 L_0}{H}(1+r_0) + \sum_{t=0}^{\infty} \frac{1}{1+R_t} w_t A_0 (1+g)^t \frac{L_0 (1+n)^t}{H}$$

$$\sum_{t=0}^{\infty} \frac{(1+g)^t (1+n)^t}{1+R_t} c_t \le k_0 (1+r_0) + \sum_{t=0}^{\infty} \frac{(1+g)^t (1+n)^t}{1+R_t} w_t$$

$$k_0 (1+r_0) + \sum_{t=0}^{\infty} \frac{(1+g)^t (1+n)^t}{1+R_t} (w_t - c_t) \ge 0 \tag{2.2}$$

Transformed technology is:

$$y_t = k_t^{\ \alpha} \tag{2.3}$$

Wages and interest rates are derivatives:

$$w_t = (1-\alpha) k_t^{\ \alpha} \tag{2.4}$$

$$r_t = \alpha k_t^{\ \alpha - 1} \tag{2.5}$$

The transformed capital accumulation is:

$$k_{t+1} A_{t+1} L_{t+1} = y_t A_t L_t + k_t A_t L_t - c_t A_t L_t$$

$$k_{t+1} \frac{A_{t+1} L_{t+1}}{A_t L_t} = y_t + k_t - c_t$$

$$k_{t+1}(1+g)(1+n) = k_t^{\ \alpha} + k_t - c_t \tag{2.8}$$

Setting up and solving the household's problem

The Langrangian for the household's problem:

$$L = B\sum_{t=0}^{\infty}\beta^t \frac{1}{1-\theta}c_t^{1-\theta} + \lambda\left\{k_0(1+r_0) + \sum_{t=0}^{\infty}\frac{(1+g)^t(1+n)^t}{1+R_t}(w_t - c_t)\right\}$$

A generic first-order condition for period *t*:

$$\frac{\partial L}{\partial c_t} = B\beta^t c_t^{-\theta} - \lambda\frac{(1+g)^t(1+n)^t}{1+R_t} = 0$$

The same thing for period *t*+1:

$$\frac{\partial L}{\partial c_t} = B\beta^{t+1}c_{t+1}^{-\theta} - \lambda\frac{(1+g)^{t+1}(1+n)^{t+1}}{1+R_{t+1}} = 0$$

Solving both for λ:

$$\lambda = \frac{B\beta^t c_t^{-\theta}(1+R_t)}{(1+g)^t(1+n)^t} = \frac{B\beta^{t+1}c_{t+1}^{-\theta}(1+R_{t+1})}{(1+g)^{t+1}(1+n)^{t+1}}$$

Simplifying gives an Euler equation:

$$\frac{c_t^{-\theta}(1+R_t)}{1} = \frac{\beta c_{t+1}^{-\theta}(1+R_{t+1})}{(1+g)(1+n)}$$

$$c_{t+1}^{\theta}(1+R_t) = \frac{\beta(1+R_{t+1})}{(1+g)(1+n)}c_t^{\theta}$$

$$c_{t+1} = \left[\frac{\beta(1+r_{t+1})}{(1+g)(1+n)}\right]^{\frac{1}{\theta}}c_t$$

Substituting β from above

$$c_{t+1} = \left[\frac{(1+r_{t+1})}{(1+\rho)(1+g)^{\theta}}\right]^{\frac{1}{\theta}}c_t$$

Substituting (2.5)

$$c_{t+1} = \left[\frac{(1+\alpha k_{t+1}^{\alpha-1})}{(1+\rho)(1+g)^{\theta}}\right]^{\frac{1}{\theta}}c_t \qquad (2.9)$$

Finding the steady state

Steady state values for *c* and *k* are found by solving the steady state versions of 2.8 and 2.9, a system of 2 equations in 2 unknowns. First solve 2.9 for the SS value of *k*.

$$\bar{c} = \left[\frac{(1+\alpha\bar{k}^{\alpha-1})}{(1+\rho)(1+g)^{\theta}}\right]^{\frac{1}{\theta}}\bar{c}$$

$$1+\alpha\bar{k}^{\alpha-1} = (1+\rho)(1+g)^{\theta}$$

$$\bar{k} = \{\tfrac{1}{\alpha}[(1+\rho)(1+g)^{\theta}-1]\}^{\frac{1}{\alpha-1}}$$

Use this along with 2.8 to solve for the SS value of *c*.

$$\bar{c} = \bar{k}^{\alpha} + \bar{k}[1-(1+g)(1+n)]$$

The saddle path

The saddle path in this case is found by taking an initial value for k_0 and substituting it into the budget constraint and (2.8). Recall the budget constraint is:

$$k_0(1+r_0) + \sum_{t=0}^{\infty}\frac{(1+g)^t(1+n)^t}{1+R_t}(w_t - c_t) = 0$$

For a given k_0 and c_0, equations (2.8) and (2.9) tell how the economy evolves over time. The saddle path is found by finding the unique value of c_0 that generates a sequence, $\{c_t, k_t\}_{t=1}^{\infty}$, which satisfies the budget constraint exactly.

An economy like this can easily be simulated using the "goal seek" function in Excel, or some other numerical method. Numerically, this can be tricky because deviating from the exact initial conditions even slightly puts us on a path that is initially close to the saddle path, but which ultimately deviates away and "explodes".

Does the SS solve the BC? (We would certainly hope so!)

$$\bar{k}(1+\bar{r}) + \sum_{t=0}^{\infty}\frac{(1+g)^t(1+n)^t}{(1+\bar{r})^t}(\bar{w}-\bar{c})$$

$$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) + \sum_{t=0}^{\infty}\frac{(1+g)^t(1+n)^t}{(1+\alpha\bar{k}^{\alpha-1})^t}[(1-\alpha)\bar{k}^{\alpha} - \bar{k}^{\alpha} - \bar{k} + (1+g)(1+n)\bar{k}]$$

$$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) + [(1-\alpha)\bar{k}^{\alpha} - \bar{k}^{\alpha} - \bar{k} + (1+g)(1+n)\bar{k}]\frac{1}{1-\frac{(1+g)(1+n)}{(1+\alpha\bar{k}^{\alpha-1})}}$$

$$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) + [(1-\alpha)\bar{k}^{\alpha} - \bar{k}^{\alpha} - \bar{k} + (1+g)(1+n)\bar{k}]\frac{(1+\alpha\bar{k}^{\alpha-1})}{(1+\alpha\bar{k}^{\alpha-1})-(1+g)(1+n)}$$

$$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) + [-\alpha\bar{k}^{\alpha} + (1+g)(1+n)\bar{k} - \bar{k}]\frac{(1+\alpha\bar{k}^{\alpha-1})}{(1+\alpha\bar{k}^{\alpha-1})-(1+g)(1+n)}$$

$$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) + \bar{k}[-\alpha\bar{k}^{\alpha-1} + (1+g)(1+n) - 1]\frac{(1+\alpha\bar{k}^{\alpha-1})}{(1+\alpha\bar{k}^{\alpha-1})-(1+g)(1+n)}$$

$\bar{k}(1+\alpha\bar{k}^{\alpha-1}) - \bar{k}(1+\alpha\bar{k}^{\alpha-1}) = 0$ It does! (sigh of relief)