Applying the Method of Undetermined Coefficients to Dynamic Stochastic General Equilibrium Models

Suppose we have a system of linear dynamic equations in our endogenous (*X*) and exogenous (*Z*) state variables which we can write as:

$$E_t\{FX_{t+1} + GX_t + HX_{t-1} + LZ_{t+1} + MZ_t\} = 0 \qquad (1)$$

X's are *S*x1
Z's are *K*x1
R equations
So, *F*, *G* & *H* are *R*x*S* and *L* & *M* are *R*x*K*

The linear laws of motion for the exogenous state variables are:

$$Z_{t+1} = NZ_t + \varepsilon_{t+1} \qquad (2)$$

N is *K*x*K* and is often diagonal, if the shocks are unrelated

We want to solve for the linear transition function

$$X_t = PX_{t-1} + QZ_t \qquad (3)$$

P is *S*x*S* and *Q* is *S*x*K*

Iterative substitution of (2) and (3) into (1) gives

$$E_t\{F[PX_t + QZ_{t+1}] + GX_t + HX_{t-1} + LZ_{t+1} + MZ_t\} = 0$$

$$F[PX_t + QNZ_t] + GX_t + HX_{t-1} + LNZ_t + MZ_t = 0$$

$$FPX_t + FQNZ_t + GX_t + HX_{t-1} + LNZ_t + MZ_t = 0$$

$$[FP + G]X_t + HX_{t-1} + [FQN + LN + M]Z_t = 0$$

$$[FP + G][PX_{t-1} + QZ_t] + HX_{t-1} + [FQN + LN + M]Z_t = 0$$

$$[(FP + G)P + H]X_{t-1} + [(FQ + L)N + (FP + G)Q + M]Z_t = 0$$

A solution requires:

$FP^2 + GP + H = 0$ (solve a matrix quadratic in *P*)

It also requires:

$$(FQ + L)N + (FP + G)Q + M = 0$$

$$FQN + LN + (FP + G)Q + M = 0$$

$$FQN + (FP + G)Q = -(LN + M)$$

$$vec\{FQN\} + vec\{(FP + G)Q\} = -vec\{(LN + M)\}$$

$$vec\{FQN\} + vec\{(FP + G)QI_k\} = -vec\{(LN + M)\}$$

Note $vec\{AXB\} = (B' \otimes A)vec\{X\}$

$$[N' \otimes F]vec\{Q\} + [I_k' \otimes (FP + G)]vec\{Q\} = -vec\{(LN + M)\}$$

$$[N' \otimes F + I_k \otimes (FP + G)]vec\{Q\} = -vec\{(LN + M)\}$$

$$vec\{Q\} = [N' \otimes F + I_k \otimes (FP + G)]^{-1}[-vec\{(LN + M)\}]$$

This only works if the first term is invertible, of course

In our simple RBC model:
$X_{t-1} = [\tilde{K} \quad \tilde{\ell}]' \equiv [\hat{K} - \overline{K} \quad \ell - \overline{\ell}]'$ and $Z_t = [y \quad z]'$
The equations to linearize are:
$$\mathscr{F}_1(\hat{K}'', \ell'', \hat{K}', \ell', \hat{K}, \ell, y', z', y, z) = \beta E\left\{e^{y}\left(\tfrac{\hat{C}}{(1+g)\hat{C}'}\right)^{\sigma}(1-\delta+r')\right\} - 1 = 0$$
$$\mathscr{F}_2(\hat{K}'', \ell'', \hat{K}', \ell', \hat{K}, \ell, y', z', y, z) = b\hat{C} - (1-\ell)\hat{w}$$
$$\hat{C} = \hat{w}\ell + (1-\delta+r)\hat{K} - (1+g+n)\hat{K}'$$
$$\hat{Y} = \hat{K}^{\alpha}(e^{z}\ell)^{1-\alpha}$$
$$\hat{w} = \tfrac{(1-\alpha)\hat{Y}}{\ell}$$
$$r = \tfrac{\alpha\hat{Y}}{\hat{K}}$$
$$\hat{C}' = \hat{w}'\ell' + (1-\delta+r')\hat{K}' - (1+g+n)\hat{K}''$$
$$\hat{Y}' = \hat{K}'^{\alpha}(e^{z'}\ell')^{1-\alpha}$$
$$\hat{w}' = \tfrac{(1-\alpha)\hat{Y}'}{\ell'}$$
$$r' = \tfrac{\alpha\hat{Y}'}{\hat{K}'}$$
We could write this as a vector function:
$$\mathscr{F}(X'', X', X, Z', Z) = \begin{bmatrix} \beta E\left\{e^{y}\left(\tfrac{\hat{C}}{(1+g)\hat{C}'}\right)^{\sigma}(1-\delta+r')\right\} - 1 \\ b\hat{C} - (1-\ell)\hat{w} \end{bmatrix}$$

Take 1st-order Taylor-series expansion of this about the SS by evaluating (4.1)
The formula is
$$\mathscr{F}(X'', X', X, Z', Z) \doteq$$
$$\mathscr{F}(\overline{X}, \overline{X}, \overline{X}, \overline{Z}, \overline{Z}) + \sum_{J \in \{X'', X', X, Z', Z\}} \frac{\partial \mathscr{F}(\overline{X}, \overline{X}, \overline{X}, \overline{Z}, \overline{Z})}{\partial J}(J - \overline{J})$$
$$\mathscr{F}(X'', X', X, Z', Z) \doteq \sum_{J \in \{X'', X', X, Z', Z\}} \frac{\partial \mathscr{F}(\overline{X}, \overline{X}, \overline{X}, 0, 0)}{\partial J}\tilde{J}$$

Numerical Derivatives

If we use $\breve{X}_i = \overline{X}(1_{S\times 1} + \varepsilon_i)$, where ε_i is an Sx1 vector of zeros with the i^{th} element replaced with the small number ε, and evaluate $\mathscr{F}$, then the result is $F(1+\varepsilon_i)$. More formally the i^{th} column of F is:
$$F_i = \frac{\partial \mathscr{F}(\overline{X}, \overline{X}, \overline{X}, 0, 0)}{\partial X_i''} = \frac{\mathscr{F}(\breve{X}_i, \overline{X}, \overline{X}, 0, 0)}{1+\varepsilon}$$
We can follow this same strategy to get G & H.

Since $\overline{y} = \overline{z} = 0$ evaluating using $Z_{t+1} = [\varepsilon \quad 0]'$ gives the first element of $L\varepsilon$. A similar strategy $Z_{t+1} = [0 \quad \varepsilon]'$ gives us the second element of L. Analogous substitutions for Z_t recover the elements of M.