Consumption under Uncertainty

Consider a household optimizing over time, where there is uncertainty. There are two different ways to model uncertainty in this context. One is more general, but more complicated. Let's look at that approach first.

Optimizing all at once in period *t*=1

Let's think of discrete uncertainty where there are a finite number of possible states of nature and each occurs with non-zero probability. Let there be I possible states of nature each period indexed by i. We will denote the state of nature in period t as $s_t \in \{1,2,...,I\}$. We will also need to keep track of the history of the economy's states and we will denote the history at date t as $H_t = (s_1, s_2, ...s_t)$.

The probability of a particular state (say i) occurring in period t, will generally depend on the particular state that existed (say j) in period t-1. We will denote this set of probabilities as $\pi_t(i \mid j)$ and note that $\sum_{j=1}^{I} \pi_t(i \mid j) = 1$. The probability of being in a given history at date t is denoted $\Pi_h = \pi_2(s_2 \mid s_1) \cdot ... \cdot \pi_t(s_t \mid s_{t-1});\ h \in \{1,2,..., H_t = I^{t-1}\}$. We will index variables in by the state history, H_t, and index this with an h subscript. Hence, the expected value of some variable, say X, in period t is given by $E\{X_t\} = \sum_{h=1}^{I^{t-1}} \Pi_h X_{ht}$.

The household maximizes expected utility at time t=1 by choosing consumption in each period and each possible state of nature. In order for this to be possible, the consumer must have access to "complete" financial markets; that is, markets where it is possible to buy and sell consumption in each period contingent on any state of nature. Arrow-Debreau (A-D) assets are a complete set and are easy to conceptualize, so let's use them. Suppose that for every possible state in every period there exists an asset that pays one unit of the consumption good if that state actually occurs and zero otherwise. We will denote the price of this asset as q_{ht}. The household is endowed with income in each period and state of nature denoted Y_{ht}, which we could view as an endowment of A-D assets. Similarly, the household chooses consumptions of C_{ht}, which we can think of as a quantity of A-D assets. Hence the budget constraint is:

$$W_0 + \sum_{t=1}^{T} \sum_{h=1}^{I^{t-1}} q_{ht} Y_{ht} \geq \sum_{t=1}^{T} \sum_{h=1}^{I^{t-1}} q_{ht} C_{ht}$$

Expected utility is given by:

$$EU = \sum_{t=1}^{T} \sum_{h=1}^{I^{t-1}} \Pi_h \left(\frac{1}{1+\rho} \right)^{t-1} u(C_{ht})$$

The generic first-order condition is:

$$\Pi_h\left(\frac{1}{1+\rho}\right)^{t-1}u'(C_{ht})-\lambda q_{ht}=0$$

Using the t and t+1 versions (where h_t is the particular history in period $_t$) gives:

$$\lambda=\frac{\Pi_{h_t}}{q_{h_t t}}\left(\frac{1}{1+\rho}\right)^{t-1}u'(C_{h_t t})=\frac{\Pi_{h_{t+1}}}{q_{h_{t+1}t+1}}\left(\frac{1}{1+\rho}\right)^{t}u'(C_{h_{t+1}t+1})$$

$$u'(C_{h_t t})=\frac{\Pi_{h_{t+1}}q_{h_t t}}{\Pi_{h_t}q_{h_{t+1}t+1}}\left(\frac{1}{1+\rho}\right)u'(C_{h_{t+1}t+1})$$

$$u'(C_{h_t t})=\frac{\pi(s_{t+1}\mid s_t)q_{h_t t}}{q_{h_{t+1}t+1}}\left(\frac{1}{1+\rho}\right)u'(C_{h_{t+1}t+1})$$, where s_t is the latest state for history h_t.

What is the ratio of $q_{h_t t}/q_{h_{t+1}t+1}$? Suppose I sold one unit of consumption in period t under state h_t. I would receive $q_{h_t t}$ units of money. I could use this to buy $q_{h_t t}/q_{h_{t+1}t+1}$ units of consumption in period t+1 if the history were h_{t+1}. However, if the history turns out to be anything other that h_{t+1}, I get nothing. This gives:

$$(1+r_{h_{t+1}t+1})=\pi(s_{t+1}\mid s_t)\frac{q_{h_t t}}{q_{h_{t+1}t+1}}$$

Substituting gives the following version of the Euler equation:

$$u'(C_{h_t t})=(1+r_{h_{t+1}t+1})\left(\frac{1}{1+\rho}\right)u'(C_{h_{t+1}t+1})$$

If we use our familiar CES utility function we can write this as:

$$C_{h_{t+1}t+1}=\left(\frac{1+r_{h_{t+1}t+1}}{1+\rho}\right)^{1/\theta}C_{h_t t}$$

We can then substitute this back into the budget constraint and use this system to solve for consumptions in every period, given the values of the q's and W_0.

Example:

Suppose T=2, and we have log utility. Normalize prices so that one unit of consumption in period 1 is worth 1, i.e $q_1=1$. Then expected utility is:

$$EU=\ln C_1+\frac{1}{1+\rho}\sum_{s=1}^{S}\pi_s\ln C_{s2}$$

and the budget constraint is:

$$W\equiv W_0+Y_1+\sum_{s=1}^{S}q_{s2}Y_{s2}\geq C_1+\sum_{s=1}^{S}q_{s2}C_{s2}$$

So the f.o.c.'s are:

$$C_1^{-1}-\lambda=0$$

$$\frac{\pi_s}{1+\rho} C_{2s}{}^{-1} - \lambda q_{s2} = 0; \ \forall s$$

Rearranging:

$$\lambda = C_1^{-1} = \frac{\pi_s C_{s2}{}^{-1}}{(1+\rho) q_{s2}}$$

or

$$C_{s2} = \left[\frac{\pi_s}{q_{s2}(1+\rho)} \right] C_1 = \left(\frac{1+r_{s2}}{1+\rho} \right) C_1$$

Substituting this back into the budget constraint:

$$W = C_1 + \sum_{s=1}^{S} q_{s2} \left[\frac{\pi_s}{q_{s2}(1+\rho)} \right] C_1 = C_1 \left(1 + \frac{1}{1+\rho} \right)$$

Solving:

$$C_1 = \frac{1+\rho}{2+\rho} W \ \text{ and } \ C_{s2} = \frac{\pi_s}{q_{s2}} \frac{1}{2+\rho} W$$

Examine this solution numerically using Excel or some other means to get an idea how consumption changes as prices and probabilities change. Note that while prices are publicly observable, the determination of probabilities is highly subjective. Consumption patterns could hence vary across individuals with otherwise identical preferences if their subjective evaluations of probabilities differ.

We could take our basic Euler equation

$$u'(C_{h_t t}) = (1+r_{h_{t+1} t+1}) \left(\frac{1}{1+\rho} \right) u'(C_{h_{t+1} t+1})$$

and multiply both sides by $\pi(s_{t+1} \mid s_t)$ and sum over s_{t+1} to get:

$$\sum_{s_{t+1}=1}^{S} \pi(s_{t+1} \mid s_t) u'(C_{h_t t}) = \sum_{s_{t+1}=1}^{S} \pi(s_{t+1} \mid s_t)(1+r_{h_{t+1} t+1}) \left(\frac{1}{1+\rho} \right) u'(C_{h_{t+1} t+1})$$

$$u'(C_{h_t t}) \sum_{s_{t+1}=1}^{S} \pi(s_{t+1} \mid s_t) = \left(\frac{1}{1+\rho} \right) \sum_{s_{t+1}=1}^{S} \pi(s_{t+1} \mid s_t)(1+r_{h_{t+1} t+1}) u'(C_{h_{t+1} t+1})$$

$$u'(C_{h_t t}) = \left(\frac{1}{1+\rho} \right) E\{(1+r_{h_{t+1} t+1}) u'(C_{h_{t+1} t+1}) \mid h_t\}$$

<u>Optimizing sequentially</u>

This approach uses a mathematical technique widely used in engineering called *dynamic programming*. We need to find a set of variables that defines the state of nature. Typically this will include exogenous variables like the level of technology and endogenous variables like savings levels or capital stocks. We will denote the exogenous state variables using a vector, A_t, and the endogenous state variables by a vector, B_t. In our context *B* will be savings, so maybe it stands for "bonds held by the household".

We assume there is some value to the household today from holding savings of B_t when the exogenous variables are A_t. The value is, of course, discounted future utility that the household expects to gain from some optimal savings and consumption plan. We denote the value function that maps A_t and B_t into "value" as $V(B_t;A_t)$. The value function is defined by the following Bellman equation:

$$V(B_t;A_t) = \underset{B_{t+1}}{Max}\left[u(C_t) + \tfrac{1}{1+\rho}E\{V(B_{t+1};A_{t+1})\}\right]$$

where the following budget constraint applies:

$$B_t(1+r_t)+Y_t = C_t + B_{t+1} \text{ or } C_t = B_t(1+r_t)+Y_t - B_{t+1}$$

and where A_t follows some exogenous stochastic process (called a *law of motion*):

$$A_{t+1} = F(A_t)$$

Note that r_t and Y_t are functions of the level of technology and capital stock which are elements of A_t.

There are two conditions that must be met.
The first is the first-order condition from the maximization problem inside the Bellman equation:

$$u'(C_t)(-1) + \tfrac{1}{1+\rho}E\{V_B(B_{t+1};A_{t+1})\} = 0$$

The second is known as he envelope condition and is found by taking the derivative of the value function with respect to B_t, holding B_{t+1} constant. Basically, this says how the value changes as B_t rises.

$$V_B(B_t;A_t) = u'(C_t)(1+r_t)$$

Combining these two gives the Euler equation:

$$u'(C_t)(-1) = \tfrac{1}{1+\rho}E\{u'(C_{t+1})(1+r_{t+1})\}$$

The goal of dynamic programming is to find a transition function, $B_{t+1} = G(B_t, A_t)$ that governs the behavior of the endogenous state variables. This can be done a variety of ways.

If the functional form of the value function is known we can work directly from the first-order condition, solving for B_{t+1}.

$$u'(B_t(1+r_t)+Y_t - B_{t+1}) = \tfrac{1}{1+\rho}E\{V_B(B_{t+1};F[A_t])\}$$

Only in a few cases do we know the form of the value function, however. The value function is derived as a solution to the Bellman equation, so it obviously depends on the functional form of the utility function. We will examine a few known solutions when we look at dynamic programming in general equilibrium.

Numerically, we can solve for the value function by iteration. Suppose we make an initial guess that the value function is $V_{(0)} = 0$, where the value in parentheses indicates the number of the iteration. We can solve the maximization problem inside the Bellman equation and get a transition function $B_{t+1} = G_{(1)}(B_t, A_t)$ which can substitute to get $V_{(1)}(B_t; A_t)$. Iterating through this process will give successive versions of the value function that eventually converge to the true value function and our transition functions will also converge to the true transition function.

In addition, there are linear or quadratic approximation techniques that can be used to get approximations to the transition functions. We will also look at some of these techniques later.

Note that the first approach is more general than dynamic programming. For example, dynamic programming is valid only for infinitely-lived agents. The sequential problem the household solves each period is identical only if the time horizon is the same in each period, which means it must be infinite. We could solve the problem for finitely-lived agents using sequential techniques, but we would need a different value function for each point in the household's life. This might work with overlapping generations models where each generation solves a similar problem at the same age, but we cannot use this when we have a single representative agent.

<u>The stochastic Euler equation</u>

Both setups above gave us the same Euler equation, namely:

$$u'(C_t) = \tfrac{1}{1+\rho} E\{u'(C_{t+1})(1+r_{t+1})\}$$

Suppose the consumer could choose between a variety of different financial assets when saving. Suppose these all have unique risk properties. The consumer's problem needs to choose not only consumption each period, but also holdings of various assets for savings each period. If we denote the return on asset *i* in period *t*+1 as r^i_{t+1}, then we get the following generic Euler equation for a given asset in a given period.

$$u'(C_t) = \tfrac{1}{1+\rho} E\{u'(C_{t+1})(1+r^i_{t+1})\}$$

Suppose we had quadratic utility (or approximated some other utility function with a quadratic) so that $u(C) = C - a\tfrac{1}{2}C^2$, $u'(C) = 1 - aC$

Recalling properties of expected values we have:

$$E\{(1-aC_{t+1})(1+r^i_{t+1})\} = E\{(1-aC_{t+1})\}E\{(1+r^i_{t+1})\} + Cov\{(1-aC_{t+1}),(1+r^i_{t+1})\}$$

$$= E\{(1-aC_{t+1})\}E\{(1+r^i_{t+1})\} - aCov\{C_{t+1}, 1+r^i_{t+1}\}$$

So,

$$1 - aC_t = \tfrac{1}{1+\rho}[E\{(1-aC_{t+1})\}E\{(1+r^i_{t+1})\} - aCov\{C_{t+1}, 1+r^i_{t+1}\}]$$

Note that variance of an asset, *per se,* does not effect the optimal choice, but rather covariance with consumption (or more correctly, marginal utility of consumption).

Risky assets may, in fact offer lower returns than riskless ones under the right conditions. To see this imagine a riskless asset where the return is known to be $\bar{r}$. Since there is no variance in the return, the covariance is zero and the Euler equation for this asset is:

$$1-aC_t = \tfrac{1}{1+\rho}E\{(1-aC_{t+1})\}(1+\bar{r})$$

Use the original Euler equation for a generic risky asset. Since the two left-hand sides are identical, the two right-hand sides are equal to each other giving us:

$$E\{r_{t+1}^i\}-\bar{r} = \frac{aCov\{C_{t+1},1+r_{t+1}^i\}}{E\{(1-aC_{t+1})\}}$$

$E\{r_{t+1}^i\}-\bar{r}$ is a risk premium for asset *i*. If the covariance of the risky asset with consumption is negative, then the return of the risky asset will be less than the riskless asset. Only if the covariance is positive does the return of the risky asset exceed that of a riskless one.

Suppose an econometrician regressed the returns of various assets on consumption growth. The coefficients from these regressions are called the "betas". The consumption CAPM model says that the betas are proportional to their premia.

The variance/covariance properties of the fundamental assets will generally be exogenous, but consumption is endogenous. The household adjusts its portfolio of assets in such a way that the resulting consumptions have covariance properties that satisfy the Euler equations for all the assets. In some cases this may involve the portfolio having negative quantities of assets held. This is interpretable as the household offering such assets for sale and agreeing to pay the purchasers the appropriate payments in the appropriate states of nature.

Consider two risky assets (say *i* & *j*) and calculate a relative risk premium under the assumption of a CES utility function.

$$C_t^{-\theta} = \tfrac{1}{1+\rho}E\{C_{t+1}{}^{-\theta}(1+r_{t+1}^i)\}$$

$$1+\rho = E\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\theta}(1+r_{t+1}^i)\right\}$$

$$1+\rho = E\left\{\left(1+g_{t+1}^C\right)^{-\theta}(1+r_{t+1}^i)\right\}$$

Which can be approximated by a Taylor-series expansion about $r_{t+1}^1 = g_{t+1}^C = 0$:

$$1+\rho = E\left\{1+r_{t+1}^i-\theta g_{t+1}^C-\theta r_{t+1}^i g_{t+1}^C+\tfrac{1}{2}\theta(\theta+1)(g_{t+1}^C)^2\right\}$$

$$\rho = E\{r_{t+1}^i\}-\theta E\{g_{t+1}^C\}-\theta E\{r_{t+1}^i g_{t+1}^C\}+\tfrac{1}{2}\theta(\theta+1)E\{(g_{t+1}^C)^2\}$$

$$\rho = E\{r_{t+1}^i\}-\theta E\{g_{t+1}^C\}-\theta[E\{r_{t+1}^i\}E\{g_{t+1}^C\}+Cov\{r_{t+1}^i,g_{t+1}^C\}]$$
$$+\tfrac{1}{2}\theta(\theta+1)[E\{(g_{t+1}^C)\}^2+Var\{g_{t+1}^C\}]$$

We can solve this for the expected return.

$$E\{r_{t+1}^i\} = \rho + \theta E\{g_{t+1}^C\} + \theta[E\{r_{t+1}^i\}E\{g_{t+1}^C\} + Cov\{r_{t+1}^i, g_{t+1}^C\}]$$
$$-\tfrac{1}{2}\theta(\theta+1)[E\{(g_{t+1}^C)\}^2 + Var\{g_{t+1}^C\}]$$

When the time period is short both the return and growth rates are small, sot we can drop the terms $E\{r_{t+1}^i\}E\{g_{t+1}^C\}$ and $E\{(g_{t+1}^C)\}^2$ as second-order small.

$$E\{r_{t+1}^i\} = \rho + \theta E\{g_{t+1}^C\} + \theta Cov\{r_{t+1}^i, g_{t+1}^C\} - \tfrac{1}{2}\theta(\theta+1)Var\{g_{t+1}^C\}$$

A relative risk premium would be the difference between the two expected returns.

$$E\{r_{t+1}^i\} - E\{r_{t+1}^j\} = \theta Cov\{r_{t+1}^i, g_{t+1}^C\} - \theta Cov\{r_{t+1}^j, g_{t+1}^C\}$$

By properties of covariances this is

$$E\{r_{t+1}^i\} - E\{r_{t+1}^j\} = \theta Cov\{r_{t+1}^i - r_{t+1}^j, g_{t+1}^C\}$$

The Equity Premium Puzzle is that if we think of *i* as being stocks and *j* as being government bonds, the relation doesn't seem to hold. The equity premium, $E\{r_{t+1}^i\} - E\{r_{t+1}^j\}$, is approximately 6%. The standard deviation of consumption growth is 3.6%, and the standard deviation of stock returns minus gov't bond returns is 16.7%, with a correlation of .4 between the two. The covariance can be found by $Corr\{x, y\} = Cov\{x, y\} / \sqrt{Var\{x\}Var\{y\}}$ and is approximately .24%. This implies a coefficient of risk aversion of $\theta = 25$, which is implausibly high.