Consumption under Certainty

Let us consider a more general case than the textbook does, where there is a subjective discount rate, ρ, and a market discount rate given by the real interest rate, r_t. Keep in mind that we are *not* doing general equilibrium in this section. We are focusing on the consumer's problem taking interest rates, taxes and income as given.

A Basic Model

Lifetime utility is

$$U = \sum_{t=1}^{T} \left(\frac{1}{1+\rho} \right)^{t-1} u(C_t)$$

Each period the budget constraint is:

$$A_{t-1}(1+r_t) + Y_t \geq C_t + T_t + A_t$$

where A denotes savings in the form of financial assets & T denotes lump-sum taxes.

The household is also restricted from being in debt at the end of the last period:

$$A_T \geq 0$$

These constraints collectively collapse to the following lifetime budget constraint:

$$(1+r_1)A_0 + \sum_{t=1}^{T} d_t (Y_t - T_t) \geq \sum_{t=1}^{T} d_t C_t \ ; \ d_t \equiv \begin{cases} 1 & \text{if } t = 1 \\ \prod_{s=2}^{t} \frac{1}{1+r_s} & \text{otherwise} \end{cases}$$

In the special case where $r_t = \bar{r} \ \forall t$ we have the following discount factor, $d_t \equiv \left(\frac{1}{1+\bar{r}} \right)^{t-1}$

Note that if we were to let the household live forever it would make no sense to impose $A_T \geq 0$, but we would impose an equivalent constraint known as the transversality condition, which is $\lim_{T \to \infty} d_T A_T = 0$. In other words, the consumer may actually have assets approaching a number that is negative, but it must be zero in present value.

Setting up the Lagrangian and taking the first-order conditions gives:

$$u'(C_t)\left(\frac{1}{1+\rho} \right)^{t-1} - \lambda d_t = 0 \ \forall t$$

Using the t and t+1 versions of the above gives:

$$\lambda = u'(C_t)\left(\frac{1}{1+\rho} \right)^{t-1} \frac{1}{d_t} = u'(C_{t+1})\left(\frac{1}{1+\rho} \right)^{t} \frac{1}{d_{t+1}} \ \text{ or } \ u'(C_t) = u'(C_{t+1}) \frac{1+r_{t+1}}{1+\rho}$$

Suppose $u(C) = \frac{C^{1-\theta}}{1-\theta}$, then this becomes:

$$C_t^{-\theta} = C_{t+1}^{-\theta} \frac{1+r_{t+1}}{1+\rho} \text{ or } C_{t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_t$$

Successive iteration gives:

$$C_t = \left[\left(\frac{1}{1+\rho}\right)^{(t-1)} \left(\frac{1}{d_t}\right)\right]^{1/\theta} C_1$$

Again, in the special case where $r_t = \bar{r} \ \forall t$ (which applies in the steady state) we have:

$$C_t = \left(\frac{1+\bar{r}}{1+\rho}\right)^{(t-1)/\theta} C_1$$

The above implies that if $\bar{r} > \rho$ consumption grows over time, if $\bar{r} < \rho$ consumption falls over time and if $\bar{r} = \rho$ consumption is constant.

Substituting our formula for C_t into the lifetime budget constraint gives:

$$W \equiv (1+r_1)A_0 + \sum_{t=1}^{T} d_t (Y_t - T_t) = \sum_{t=1}^{T} d_t^{\frac{1-\theta}{\theta}} \left(\frac{1}{1+\rho}\right)^{(t-1)/\theta} C_1$$

Since this is too complicated, impose $r_t = \bar{r} \ \forall t$ and $\theta = 1$ to get:

$$W = \sum_{t=1}^{T} \left(\frac{1}{1+\rho}\right)^{t-1} C_1 = C_1 \left[\sum_{t=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{t-1} - \sum_{t=T+1}^{\infty} \left(\frac{1}{1+\rho}\right)^{t-1}\right]$$

$$W = C_1 \left[\frac{1}{1-\frac{1}{1+\rho}} - \left(\frac{1}{1+\rho}\right)^{T} \frac{1}{1-\frac{1}{1+\rho}}\right] = C_1 \left[1 - \left(\frac{1}{1+\rho}\right)^{T}\right] \left(\frac{1+\rho}{\rho}\right)$$

Solving for C_1 gives:

$$C_1 = \left(\frac{\rho}{1+\rho}\right) \left(\frac{(1+\rho)^T}{(1+\rho)^T - 1}\right) W$$

Note that if we take the limit as T approaches infinity we get:

$$C_1 = \left(\frac{\rho}{1+\rho}\right) W$$

C_t is:

$$C_t = \left(\frac{1+\bar{r}}{1+\rho}\right)^{t-1} \left(\frac{\rho}{1+\rho}\right) \left(\frac{(1+\rho)^T}{(1+\rho)^T - 1}\right) W$$

Numerical Simulation of More Complicated Models

Think about solving this model numerically, perhaps when the utility function is not so cooperative, when interest rates are not constant, or when taxes are non-linear function of consumption or some other endogenous variable.

Given a sequence of incomes, $\{Y_t\}_{t=1}^{T}$, an initial wealth, A_0, a sequence of interest rates, $\{r_t\}_{t=1}^{T}$, and a momentary utility function, $u(C)$, we could implement the following strategy.

1. Choose an initial guess for first-period consumption, C_1.
2. Using the Euler equation, solve for the entire sequence of consumptions, $\{C_t\}_{t=1}^{T}$.
3. Using the single-period budget constraint, solve for the sequence of savings levels, $\{A_t\}_{t=1}^{T}$.
4. Use numerical techniques to find the unique value for initial consumption that satisfies the budget constraint with equality, i.e. that sets $A_T = 0$

Here is an example:

Interest rates are not constant. Momentary utility is given by $u(C) = \dfrac{C^{1-\theta}}{1-\theta}$ The government taxes interest earnings ($r_t A_{t-1}$), other income (Y_t) and consumption, all at different rates.

We have marginal utility given by, $u'(C) = C^{-\theta}$

The single-period budget constraint is:

$$A_{t-1}[1+(1-\tau_r)r_t] + (1-\tau_Y)Y_t = (1-\tau_C)C_t + A_t$$

The lifetime budget constraint is:

$$[1+(1-\tau_r)r_1]A_0 + \sum_{t=1}^{T} d_t(1-\tau_Y)Y_t \geq \sum_{t=1}^{T} d_t(1-\tau_C)C_t \ ; \ d_t \equiv \begin{cases} 1 & \text{if } t=1 \\ \displaystyle\prod_{s=2}^{t} \frac{1}{1+(1-\tau_r)r_s} & \text{otherwise} \end{cases}$$

The generic first-order condition is:

$$u'(C_t)\left(\frac{1}{1+\rho}\right)^{t-1} - \lambda d_t(1-\tau_C) = 0$$

So the Euler equation is:

$$u'(C_t) = \frac{1+(1-\tau_r)r_{t+1}}{1+\rho} u'(C_{t+1})$$

Imposing our functional form and solving yields:

$$C_{t+1} = \left(\frac{1 + (1 - \tau_r) r_{t+1}}{1 + \rho} \right)^{1/\theta} C_t$$

Note that if there were no closed form solution, we could still solve for C_{t+1} numerically, given C_t.

The single-period budget constraint gives:

$$A_t = A_{t-1}[1 + (1 - \tau_r) r_t] + (1 - \tau_Y) Y_t - (1 - \tau_C) C_t$$

Given a starting A_0, we use the two equations above to find the whole sequence of C's and A's. The numerical solution is found by searching over the values of C_1 to find the one that gives A_T=0.

An MS Excel spreadsheet that implements this strategy is available on Blackboard.

Extensions to Think About

A) Suppose the consumer faced a non-zero probability of death each period, but there was otherwise no uncertainty. Further suppose that utility in the case of death is zero. What does the Euler equation look like in this case? Suppose the probability of death is not constant, but rises with age. How does it alter the simulation?

B) Suppose the consumer's discount rate, ρ, was time varying. Micro studies have shown that concern about the future rises as consumption levels rise. It may also change with age. How would the simulation look if ρ_t fell with age?

C) Some studies find that consumption rises as workers get older until their mid-fifties and falls thereafter. If interest rates are constant over time, how can we modify our model to generate this result?