Brigham Young University

Economics 581 – Advanced Macroeconomics

Fall Semester 2007

Final Exam key

December 18, 2006

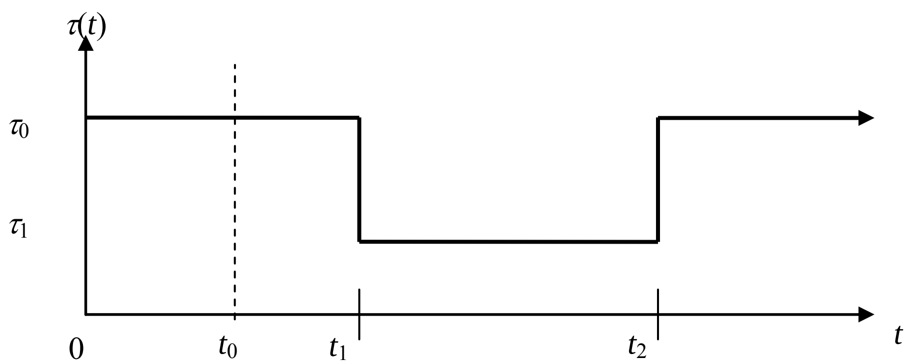
This exam is open book and open notes. You may attach (clearly written & labeled, easily legible) notes to this exam to show your work.

1. (20 points) Consider the continuous-time Ramsey-Cass-Koopmans model with taxation from the textbook and homework with the following dynamic equations:

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau(t))f'(k(t)) - \rho - \theta g}{\theta}$$

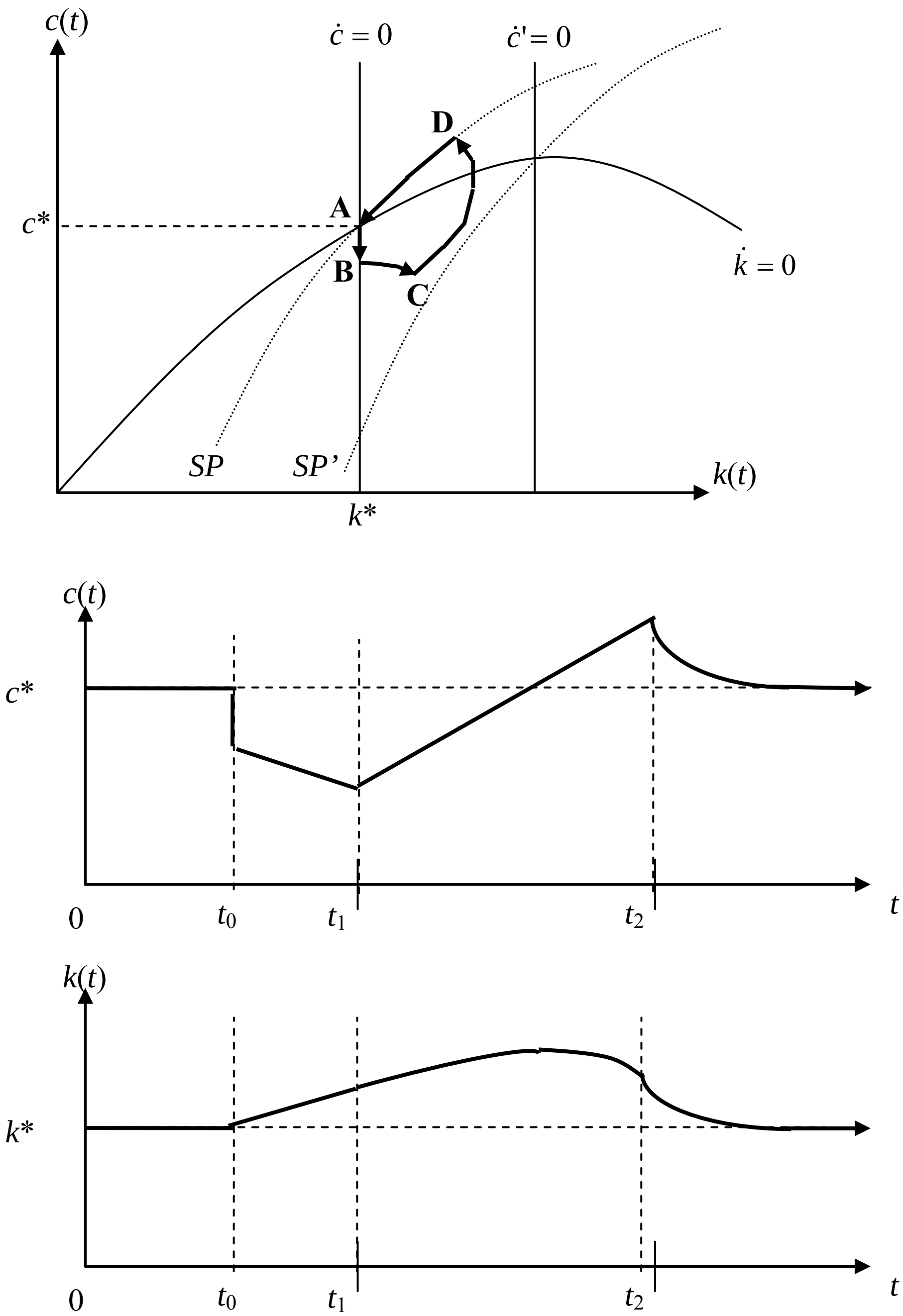
$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

Using the phase diagram illustrate the effects of the following time-path for the tax rate. The tax rate is initially at τ_0, but at time t_0 it is announced that it will fall to τ_1 at time t_1 and then rise back to τ_0 at time t_2. Assume the economy is in the steady state prior to time t_0.



Show the behavior of $c(t)$ and $k(t)$ over time in graphs as well as illustrating the time-path in the phase diagram.

Work this problem out backwards through time. We know in the long run we are at (k^*,c^*). We also know we approach this asymptotically along the saddle path (*SP*) from time t_2 to infinity [Movement from point D to point A]. In addition we know that we can only change $c(t)$ discretely at t_0 when the new information is revealed. After that, $c(t)$ must change continuously over time. From t_1 to t_2, the time path is governed by dynamics with a tax rate of τ_1 [Movement from point C to point D]. From t_0 to t_1, the time path is governed by dynamics with a tax rate of τ_0 [Movement from point B to point C]. At point t_0 the value of $c(t)$ jumps discontinuously, $k(t)$ cannot move however. [Movement from point A to point B]



c(t)
ċ = 0
ċ' = 0
D
A
c*
B
C
k̇ = 0
SP
SP'
k(t)
k*
c(t)
c*
0
t0
t1
t2
t
k(t)
k*
0
t0
t1
t2
t

2. Consider our discrete-time real business cycle (RBC) model from the notes in class. Suppose there are no y shocks. The system of equations that describe this economy is:

$z' = \rho z + \varepsilon_z'$; where $\varepsilon_z' \sim iid(0, \sigma_z^2)$

$1 = \beta E\{(\frac{C}{C'})^{\sigma}(1 - \delta + r')\}$

$C = wL + (1 - \delta + r)K - K'$

$Y = K^{\alpha}(e^{gt+z}L)^{1-\alpha}$

$wL = (1 - \alpha)Y$

$rK = \alpha Y$

Part A (10 points) Imagine now that the government imposes a constant income tax rate (τ) and uses all the proceeds to purchase of goods and services (G) each period. Further suppose that G enters the household utility function as follows:

$u(C) = \frac{1}{1-\sigma}C^{1-\sigma} + \kappa\frac{1}{1-\gamma}G^{1-\gamma}$

Write down the household's problem and solve it to get the intertemporal Euler equation.

$V(K, \mathbf{\Omega}) = \underset{K'}{Max}\, u(C) + \beta E\{V(K', \mathbf{\Omega}')\}$

$C = (1 - \tau)(wL + rK) + (1 - \delta)K - K'$

The first-order condition is:

$C^{-\sigma}(-1) + \beta E\{V_K(K', \Omega')\} = 0$

The envelope condition is.

$V_K(K, \Omega) = C^{-\sigma}[1 - \delta + (1 - \tau)r]$

Combining andthese gives the following Euler equation:

$$C^{-\sigma} = \beta E\{(C')^{-\sigma}[1 - \delta + (1 - \tau)r']\}$$

Part B (10 points) Write down the system of equations corresponding to the six equations above.

$z' = \rho z + \varepsilon_z'$**; where** $\varepsilon_z' \sim iid(0, \sigma_z^2)$

$1 = \beta E\{(\frac{C}{C'})^{\sigma}[1 - \delta + (1 - \tau)r']\}$

$C = (1 - \tau)(wL + rK) + (1 - \delta)K - K'$

$Y = K^{\alpha}(e^{gt+z}L)^{1-\alpha}$

$wL = (1 - \alpha)Y$

$rK = \alpha Y$

$G = \tau Y$

Part C (10 points) Transform these equations to their stationary versions. (Assume $0<\rho<1$). Describe the steady state for this economy. Give formulas for the steady state value of capital and the interest rate ($\overline{K}\ \&\ \overline{r}$) as a functions of the parameters of the model ($\beta, \sigma, g, \delta, \alpha, \tau\ \&\ L$). Also give formulas for $\overline{Y}, \overline{w}, \overline{C}\ \&\ \overline{G}$ as functions of these same parameters and $\overline{K}\ \&\ \overline{r}$.

$\bar{z} = 0$

$1 = \beta(1+g)^{-\sigma}[1-\delta+(1-\tau)\bar{r}]$

$\bar{C} = (1-\tau)(\bar{w}L+\bar{r}\bar{K})+(1-\delta)\bar{K}-(1+g)\bar{K}$

$\bar{Y} = \bar{K}^{\alpha}L^{1-\alpha}$

$\bar{w}L = (1-\alpha)\bar{Y}$

$\bar{r}\bar{K} = \alpha\bar{Y}$

$\bar{G} = \tau\bar{Y}$

Solve the first equation for $\bar{r}$:

$$\boxed{\bar{r} = \frac{(1+g)^{\sigma}+\beta\delta-\beta}{\beta(1-\tau)}}$$

Using the 4th and 6th and solving for $\bar{K}$:

$$\bar{r} = \alpha\bar{K}^{\alpha-1}L^{1-\alpha} \qquad \textbf{or} \qquad \bar{K} = \left(\frac{\alpha}{\bar{r}}\right)^{\frac{1}{1-\alpha}}L$$

Substituting $\bar{r}$ from above

$$\boxed{\bar{K} = \left(\frac{\beta\alpha(1-\tau)}{(1+g)^{\sigma}+\beta\delta-\beta}\right)^{\frac{1}{1-\alpha}}L}$$

$$\boxed{\begin{aligned}&\bar{Y} = \bar{K}^{\alpha}L^{1-\alpha}\\&\bar{w} = (1-\alpha)\bar{K}^{\alpha}L^{-\alpha}\\&\bar{C} = (1-\tau)\bar{w}L+[(1-\tau)\bar{r}-\delta-g]\bar{K}\\&\bar{G} = \tau\bar{K}^{\alpha}L^{1-\alpha}\end{aligned}}$$

Part D (10 points) State the effects of raising τ on $\bar{r},\bar{K},\bar{Y},\bar{w},\bar{C}$ & $\bar{G}$ and lifetime steady state utility.

$$\bar{r} = \frac{(1+g)^{\sigma}+\beta\delta-\beta}{\beta(1-\tau)}$$

As τ rises the steady state value of the interest rate also rises.

$$\bar{K} = \left(\frac{\beta\alpha(1-\tau)}{(1+g)^{\sigma}+\beta\delta-\beta}\right)^{\frac{1}{1-\alpha}}L$$

As τ rises the steady state value of the capital stock falls.

$$\bar{Y} = \left(\frac{\beta\alpha(1-\tau)}{(1+g)^{\sigma}+\beta\delta-\beta}\right)^{\frac{\alpha}{1-\alpha}}L$$

As τ rises the steady state value of output falls.

$$\bar{w} = (1-\alpha)\left(\frac{\beta\alpha(1-\tau)}{(1+g)^{\sigma}+\beta\delta-\beta}\right)^{\frac{\alpha}{1-\alpha}}$$

As τ rises the steady state value of the wage falls.

$$\overline{C} = (1-\tau)^{\frac{1}{1-\alpha}} L \left[\left(\frac{\beta\alpha}{(1+g)^{\sigma} + \beta\delta - \beta} \right)^{\frac{\alpha}{1-\alpha}} + \left(\frac{(1+g)^{\sigma} - (1+g)\beta}{\beta} \right) \left(\frac{\beta\alpha}{(1+g)^{\sigma} + \beta\delta - \beta} \right)^{\frac{1}{1-\alpha}} \right]$$

As τ rises the steady state value of consumption falls.

$$\overline{G} = \tau(1-\tau)^{\frac{\alpha}{1-\alpha}} \Gamma^{\frac{\alpha}{1-\alpha}} L \textbf{;} \ \Gamma \equiv \frac{\beta\alpha}{(1+g)^{\sigma} + \beta\delta - \beta}$$

$$\frac{\partial \overline{G}}{\partial \tau} = \left[(1-\tau)^{\frac{\alpha}{1-\alpha}} - \tau \tfrac{\alpha}{1-\alpha} (1-\tau)^{\frac{\alpha}{1-\alpha}-1} \right] \Gamma^{\frac{\alpha}{1-\alpha}} L$$

$$\frac{\partial \overline{G}}{\partial \tau} = (1-\tau)^{-\frac{1}{1-\alpha}} (1 - \tfrac{1-2\alpha}{1-\alpha} \tau) \Gamma^{\frac{\alpha}{1-\alpha}} L$$

As τ rises the steady state value of government spending rises.
Lifetime SS utility is given by:

$$\overline{U} = \sum_{t=0}^{\infty} \beta^{t} \{ \tfrac{1}{1-\sigma} [\overline{C}(1+g)^{t}]^{1-\sigma} + \kappa \tfrac{1}{1-\gamma} [\overline{G}(1+g)^{t}]^{1-\gamma} \}$$

$$\overline{U} = \tfrac{1}{1-\sigma} \overline{C}^{1-\sigma} \sum_{t=0}^{\infty} [\beta(1+g)^{1-\sigma}]^{t} + \kappa \tfrac{1}{1-\gamma} \overline{G}^{1-\sigma} \sum_{t=0}^{\infty} [\beta(1+g)^{1-\gamma}]^{t}$$

$$\overline{U} = \frac{\overline{C}^{1-\sigma}}{(1-\sigma)[1-\beta(1+g)^{1-\sigma}]} + \frac{\kappa \overline{G}^{1-\sigma}}{(1-\gamma)[1-\beta(1+g)^{1-\gamma}]}$$

As τ rises the utility from private consumption falls. However, utility from government purchases rises. Which effect is bigger depends critically on the value of κ.

3. (20 points) Consider a model of endogenous growth like the one in chapter 3 of the text where there is no capital. However, assume that there are two types of capital. These are: Applied knowledge, denoted $A(t)$ as in the text. And basic knowledge, denoted $B(t)$. a_L of the available labor force is used in applied R&D, and b_L is used in basic R&D. The rest is used in production of goods.

Production is given by: $Y(t) = A(t)(1 - a_L - b_L)L(t)$
Applied knowledge evolves according to: $\dot{A}(t) = D[a_L L(t)]^{\gamma} A(t)^{\theta} B(t)^{\phi}$
Basic knowledge evolves according to: $\dot{B}(t) = E[b_L L(t)]^{\lambda} B(t)^{\kappa} A(t)^{\mu}$
Labor grows exogenously: $\dot{L}(t) = nL(t)$

Solve this model for the steady state growth rates of $A, B \& Y$. Show what restrictions on the parameter values are necessary to ensure that there is a steady state.

Calculate the instantaneous growth rates of *A* & *B*

$$g_A(t) \equiv \frac{\dot{A}(t)}{A(t)} = D[a_L L(t)]^{\gamma} A(t)^{\theta-1} B(t)^{\phi}$$

$$g_B(t) \equiv \frac{\dot{B}(t)}{B(t)} = E[b_L L(t)]^{\lambda} B(t)^{\kappa-1} A(t)^{\mu}$$

Take logs and then time derivatives of the above:

$$\frac{\dot{g}_A(t)}{g_A(t)} = \gamma n + (\theta - 1)g_A(t) + \phi g_B(t)$$

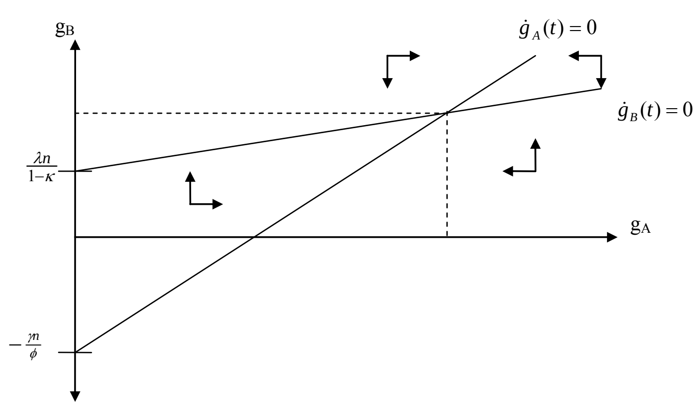
$$\frac{\dot{g}_B(t)}{g_B(t)} = \lambda n + (\kappa - 1)g_B(t) + \mu g_A(t)$$

Set these equal to zero and put these loci in slope-intercept form:

$$\overline{g}_B = -\tfrac{\gamma n}{\phi} + \tfrac{1-\theta}{\phi}\overline{g}_A$$

$$\overline{g}_B = \tfrac{\lambda n}{1-\kappa} + \tfrac{\mu}{1-\kappa}\overline{g}_A$$

Plot these in a phase diagram:



In order for the growth of *A* & *B* to be finite in the steady state we need the slope of the *B* locus to be less than the slope of the *A* locus.

$$\frac{1-\theta}{\phi} > \frac{\mu}{1-\kappa} \textbf{ or } (1-\theta)(1-\kappa) > \phi\mu$$

We can solve this system algebraically:

$$\tfrac{\lambda n}{1-\kappa} + \tfrac{\gamma n}{\phi} = (\tfrac{1-\theta}{\phi} - \tfrac{\mu}{1-\kappa})\overline{g}_A$$

$$\overline{g}_A = \frac{\frac{\lambda n}{1-\kappa} + \frac{\gamma n}{\phi}}{\frac{1-\theta}{\phi} - \frac{\mu}{1-\kappa}} = \frac{\frac{\lambda n\phi + (1-\kappa)\gamma n}{(1-\kappa)\phi}}{\frac{(1-\theta)(1-\kappa)-\phi\mu}{(1-\kappa)\phi}} = \left\{\frac{\lambda\phi + (1-\kappa)\gamma}{(1-\theta)(1-\kappa) - \phi\mu}\right\} n$$

Putting this back into $\overline{g}_B = -\frac{\gamma n}{\phi} + \frac{1-\theta}{\phi}\overline{g}_A$**:**

$$\overline{g}_B = \frac{\lambda\phi + (1-\kappa)\gamma}{\phi(1-\kappa-\mu)} n - \frac{\gamma}{\phi} n$$

$$\overline{g}_B = \left\{\frac{\lambda\phi + \gamma\mu}{\phi(1-\kappa-\mu)}\right\} n$$

Taking the log and time derivative of the 1st equation:

$$g_Y(t) = g_A(t) + n$$

Substituting into the above:

$$\overline{g}_Y = \left\{\frac{\lambda\phi + (1-\kappa)\gamma}{(1-\theta)(1-\kappa) - \phi\mu} + 1\right\} n$$

$$\overline{g}_Y = \left\{\frac{\lambda\phi + (1-\kappa)\gamma + (1-\theta)(1-\kappa) - \phi\mu}{(1-\theta)(1-\kappa) - \phi\mu}\right\} n$$

$$\overline{g}_Y = \left\{\frac{\phi(\lambda-\mu) + (1-\kappa)(\gamma + 1 - \theta)}{(1-\theta)(1-\kappa) - \phi\mu}\right\} n$$