**Brigham Young University Department of Economics**

# Economics 459 - International Monetary Theory

***Derivation of the Risk Premium in Covered vs Uncovered Investments***

Let the one-period investor maximize utility from consuming using next period's wealth. Let utility take the following form: $U(C) = \ln C$.

Let the returns on all potential assets be distributed normally, i.e. $r_i \sim N(\mu_i, \sigma_i^2)$.

The investor starts off with initial wealth of $S_0$.

Hence consumption next period is $C = S_0 \sum_{i=1}^{I} w_i (1 + r_i)$

Which we can rewrite as follows:

$$C = S_0 \sum_{i=1}^{I} w_i (1 + r_i) = S_0 \left( 1 + \sum_{i=1}^{I} r_i w_i \right)$$

The investor's problem is:

$$\underset{w_i}{Max} \quad U = E\{U(C)\}; \;\; s.t. \; \sum_{i=1}^{I} w_i = 1$$

With these this utility function and normal distributions for the returns, we can rewrite a utility maximization problem as follows:

$$\underset{w_i}{Max} \quad U = E\{\ln C\} - \tfrac{\gamma}{2} V\{\ln C\}; \;\; s.t. \; \sum_{i=1}^{I} w_i = 1; \; \ln C = \ln S_0 + \sum_{i=1}^{I} w_i r_i$$

Taking the appropriate expected values of ln*C* gives:

$$E\{\ln C\} = \sum_{i=1}^{I} w_i \mu_i \qquad\qquad V\{\ln C\} = \sum_{i=1}^{I} \sum_{j=1}^{I} w_i w_j \sigma_{ij} \; ; \; (\sigma_{ii} = \sigma_i^2)$$

Hence, the Lagrangian for this problem is given by:

$$L = \sum_{i=1}^{I} w_i \mu_i - \tfrac{\gamma}{2} \left( \sum_{i=1}^{I} \sum_{j=1}^{I} w_i w_j \sigma_{ij} \right) + \lambda \left( \sum_{i=1}^{I} w_i - 1 \right)$$

A generic first-order condition (with respect to element *n*) is:

$$\mu_n - \gamma \sum_{i=1}^{I} w_i \sigma_{in} - \lambda = 0$$

Note given the construction of ln*C*, the second term is the covariance of asset *n* with ln*C*.

Suppose the return on asset number *S* is riskless. Then all the covariance terms ($\sigma_{iS}$) will be zero. And we get $\mu_S = \lambda$.

Substituting this back into the generic first-order condition gives:

$$\mu_n - \mu_S = \gamma \sum_{i=1}^{I} w_i \sigma_{in} = \gamma Cov\{r_n, \ln C\} \tag{1}$$

The above is the "excess return" or "absolute risk premium", i.e the additional return risky asset *n* must offer above the risk-free rate of return.

A "relative risk premium" between two assets which are both risky, say asset *n* and asset *m* is given by:

$$\mu_n - \mu_m = \gamma \sum_{i=1}^{I} w_i \sigma_{in} - \gamma \sum_{i=1}^{I} w_i \sigma_{im} = \gamma Cov\{r_n - r_m, \ln C\} \equiv \rho_{mn} \tag{2}$$

Note that these conditions are necessary for investors to be maximizing their objective functions. The way they ensure that these conditions are met is by adjusting the weights in their portfolios, the $w_i$'s

Now, put some international structure on the assets. Use the following definitions:

Home country nominal interest rate - $i_h \sim rv(\mu_h, \sigma_h^2)$

Foreign country nominal interest rate - $i_f \sim rv(\mu_f, \sigma_f^2)$

Home country inflation rate - $\pi \sim rv(\mu_\pi, \sigma_\pi^2)$

Appreciation rate in spot exchange rate - $\varepsilon' \sim rv(\mu_\varepsilon, \sigma_\varepsilon^2)$

Forward premium - $\phi$, a constant

Covariances between these random variables may not be zero and are denoted as $\sigma_{xy}$; $x, y \in \{h, f, \pi, \varepsilon\}$

First, put all the returns in real terms, i.e divide by the rate of inflation. Let there be a domestic strategy denoted with a *D*.

$$1 + r_D = \frac{1 + i_h}{1 + \pi} \quad \text{or} \qquad r_D = i_h - \pi$$

Next, let there be a covered foreign strategy and an uncovered foreign strategy, denoted with and *C* & *U*.

$$1 + r_C = \frac{f}{s}\frac{1 + i_f}{1 + \pi} \quad \text{or} \qquad r_C = \phi + i_f - \pi$$

$$1 + r_U = \frac{s'}{s}\frac{1 + i_f}{1 + \pi} \quad \text{or} \qquad r_U = \varepsilon' + i_f - \pi$$

Taking expected values:

$$\mu_D \equiv E\{r_D\} = E\{i_h\} - E\{\pi\}$$
$$\mu_C \equiv E\{r_C\} = \phi + E\{i_f\} - E\{\pi\}$$
$$\mu_U \equiv E\{r_U\} = E\{\varepsilon'\} + E\{i_f\} - E\{\pi\}$$

Taking Covariances:

$$Cov\{r_D, \ln C\} = Cov\{i_h, \ln C\} - Cov\{\pi, \ln C\}$$
$$Cov\{r_D, \ln C\} = Cov\{i_f, \ln C\} - Cov\{\pi, \ln C\}$$
$$Cov\{r_D, \ln C\} = Cov\{i_f, \ln C\} + Cov\{\varepsilon', \ln C\} - Cov\{\pi, \ln C\}$$

The relative risk premium between the domestic strategy and the covered foreign one is:

$$\rho_{DC} = \mu_D - \mu_C = \gamma \sum_{i=1}^{I} w_i \sigma_{iD} - \gamma \sum_{i=1}^{I} w_i \sigma_{iC} = \gamma Cov\{r_D - r_C, \ln C\}$$
$$\rho_{DC} = \gamma Cov\{(i_h - \pi) - (\phi + i_f - \pi), \ln C\}$$
$$\rho_{DC} = \gamma Cov\{i_h - i_f, \ln C\} = \gamma[Cov\{i_h, \ln C\} - Cov\{i_f, \ln C\}]$$

Note that under the assumption that the domestic and foreign nominal interest rates are certain ($\sigma_h^2 = \sigma_f^2 = 0$) we have $\rho_{DC} = 0$. This is because the covariances of $i_h$ and $i_f$ with any random variable (including $\ln C$) are zero.

Also note that under the assumption that they are uncertain but have identical variance/covariance properties ($\sigma_{hx} = \sigma_{fx}$, for any random variable *x*, including $\ln C$), we also have $\rho_{DC} = 0$.

In both of these cases we can derive our covered interest rate parity (CIRP) condition. We have $0 = \rho_{DC} = \mu_D - \mu_C = E\{i_h\} - E\{\pi\} - \phi - E\{i_f\} + E\{\pi\}$.
Which reduces to:

$$E\{i_h\} - E\{i_f\} = \phi \qquad (3)$$

The relative risk premium between the domestic strategy and the uncovered foreign one is:

$$\rho_{DU} = \mu_D - \mu_U = \gamma \sum_{i=1}^{I} w_i \sigma_{iD} - \gamma \sum_{i=1}^{I} w_i \sigma_{iU} = \gamma Cov\{r_D - r_U, \ln C\}$$
$$\rho_{DU} = \gamma Cov\{(i_h - \pi) - (\varepsilon' + i_f - \pi), \ln C\}$$
$$\rho_{DU} = \gamma Cov\{i_h - \varepsilon' - i_f, \ln C\} = \gamma[Cov\{i_h, \ln C\} - Cov\{\varepsilon', \ln C\} - Cov\{i_f, \ln C\}]$$

Note that under the assumptions we made for the covered case we have $\rho_{DU} = -\gamma Cov\{\varepsilon', \ln C\}$.

This gives us the following modified uncovered interest rate parity (UIRP) condition:

$$E\{i_h\} - E\{i_f\} = E\{\varepsilon'\} - \gamma Cov\{\varepsilon', \ln C\} \qquad (4)$$

A rational expectations assumption is:

$$\varepsilon' = E\{\varepsilon'\} + e';\ e' \sim rv(0, \sigma_e^2) \qquad (5)$$

Combining (3) – (5) gives:

$$\varepsilon' = \phi - \gamma Cov\{\varepsilon', \ln C\} + e' \qquad (6)$$

Now consider the typical forward bias regression:

$$\varepsilon' = \alpha + \beta\phi + u' \qquad (7)$$

The risk premium, $-\gamma Cov\{\varepsilon', \ln C\}$, is unobservable and is not included in our regression in (7).

If this risk premium is constant, then the correspondence between (6) and (7) gives:

$$\alpha = -\gamma Cov\{\varepsilon', \ln C\}$$
$$\beta = 1$$
$$u' = e'$$

Hence a constant risk premium cannot explain the bias we observe, where estimates of $\beta$ are negative.

If it is time-varying, then the correspondence between (6) and (7) gives:

$$\alpha = 0$$
$$\beta = 1$$
$$u' = e' - \gamma Cov\{\varepsilon', \ln C\}$$

As long as $-\gamma Cov\{\varepsilon', \ln C\}$ is uncorrelated with $\phi$, then we can view it as an additional source of error in our regression, but it won't bias our estimate of $\beta$. However, if it is correlated with $\phi$, then our estimate of $\beta$ could be biased. To see this imaging that whenever $\phi$ is high, the unobserved risk premium is low. Then, even though $\beta$ is really 1 and the rise in $\phi$ causes $\varepsilon'$ to go up one-for-one, we also have a lower value for the unobserved risk premium which forces $\varepsilon'$ lower. Since we don't observe the risk premium, it appears as if $\varepsilon'$ is not rising one-for-one. That is, the estimate of $\beta$ that we get from a regression is biased downward. If the correlation is strong enough the unobserved effect could outweigh the observed effect and we could get an estimate that is negative, even though the true effect is really $\beta = 1$.

Do we have reason to believe that $-\gamma Cov\{\varepsilon', \ln C\}$ is negatively correlated with $\phi$?

Note that $\gamma Cov\{\varepsilon', \ln C\}$ could vary because $\gamma$ varies over time, or because $Cov\{\varepsilon', \ln C\}$ does. Recall that $\gamma$ is the degree of risk aversion. Normally, we might expect that there is no reason for either $\gamma$ or $Cov\{\varepsilon', \ln C\}$ to change ways that are correlated with $\phi$.

However, suppose the economy is in a recession so that $\ln C$ is expected to fall dramatically. It may be that people become more risk averse during recessions. In this case $\gamma$ would rise. At the same time the central bank might use monetary policy to stimulate the economy. They could print a great deal of domestic currency, which would lead people to expect the foreign currency to gain value, the forward premium to rise. Hence $\phi$ and $-\gamma Cov\{\varepsilon', \ln C\}$ would be negatively correlated and our estimate of $\beta$ would be biased downward.

In other words, it is possible that the behavior of the central bank in driving the anomaly we have observed.