Brigham Young University Department of Economics

Economics 459 - International Monetary Theory

Proof that bubbles formula is a solution to the asset pricing formula

Recall the asset pricing formula is:

$$p_t = d_t + \frac{1}{1+i} E_t\{p_{t+1}\} \tag{1}$$

The bubbles solution is:

$$p_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^s E_t\{d_{t+s}\} + (1+i)^t a \tag{2}$$

The first term on the right-hand-side (the sum) is the fundamentals term
The second term is the bubbles term which we will denote b_t.

In order for (2) to be a solution to (1) we must be able to substitute the appropriate variations of (2) into (1) and still have the equation hold.

To do this, first update (2) to period t+1 and take expectations as of date t:
This gives:

$$E_t\{p_{t+1}\} = \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^s E_t\{d_{t+1+s}\} + (1+i)^{t+1} a \tag{3}$$

now substitute (3) into (1) to get:

$$p_t = d_t + \frac{1}{1+i}\left[\sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^s E_t\{d_{t+1+s}\} + (1+i)^{t+1} a\right]$$

simplifying a little:

$$p_t = d_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^{s+1} E_t\{d_{t+1+s}\} + (1+i)^t a \tag{4}$$

Now note that the first two terms on the right-hand-side of (4) can be rewritten:

$$d_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^{s+1} E_t\{d_{t+1+s}\} = d_t + \frac{1}{1+i} E_t\{d_{t+1}\} + \left(\frac{1}{1+i}\right)^2 E_t\{d_{t+2}\} + \left(\frac{1}{1+i}\right)^3 E_t\{d_{t+3}\} + \ldots$$

$$= \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^s E_t\{d_{t+s}\}$$

so that (4) reduces to:

$$p_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+i}\right)^s E_t\{d_{t+s}\} + (1+i)^t a$$

Hence, since substituting (2) into both the right and left-hand side of (1) still results in (1) holding true, (2) is a solution to equation (1).

Note that the only real restriction on the bubbles term is that the expected value of a be constant so that the bubble term is growing by (1+i) each period. Many types of stochastic processes satisfy this restriction.

The most common assumption is that the bubble will burst with some constant probability, i.e.:

$$b_t = \begin{cases} \frac{1+i}{\pi} b_{t-1} & \text{with probability } \pi \\ 0 & \text{with probability } 1-\pi \end{cases} \qquad (5)$$

Note that the bubble term could consist of several terms like (5) summed together, leading to multiple bubbles. Note also that the probabilities could vary over time instead of remaining constant.

$$p_t = \sum_{s=0}^{T} \left(\tfrac{1}{1+i}\right)^s E_t\{d_{t+s}\} + (1+i)^t a_{t+s};\ E_t\{a_{t+s}\} = \overline{a}$$

In an international finance context:

$d_{t+s} = d^h_{t+s}$ for a domestic investment

$d_{t+s} = \frac{s_{t+s}}{s_t} d^f_{t+s}$ for an uncovered foreign investment

and

$d_{t+s} = \frac{f^s_t}{s_t} d^f_{t+s}$ for a covered foreign investment

these give:

$$p^d_t = \sum_{s=0}^{T} \left(\tfrac{1}{1+i}\right)^s E_t\{d^h_{t+s}\} + (1+i)^t a^d_{t+s};\ E_t\{a^d_{t+s}\} = \overline{a}^d$$

$$p^u_t = \sum_{s=0}^{T} \left(\tfrac{1}{1+i}\right)^s \tfrac{1}{s_t} E_t\{s_{t+s} d^f_{t+s}\} + (1+i)^t a^u_{t+s};\ E_t\{a^u_{t+s}\} = \overline{a}^u$$

$$p^c_t = \sum_{s=0}^{T} \left(\tfrac{1}{1+i}\right)^s \tfrac{f^s_t}{s_t} E_t\{d^f_{t+s}\} + (1+i)^t a^c_{t+s};\ E_t\{a^c_{t+s}\} = \overline{a}^c$$