

OLG Life Cycle Models: New Solution Methods and Applications *

Richard W. Evans[†]
Brigham Young University
revans@byu.edu

Kerk L. Phillips[‡]
Brigham Young University
kerk_phillips@byu.edu

PRELIMINARY

May 2009

(version 09.05.a)

Abstract

The overlapping generations (OLG) model is an important framework for analyzing any type of question in which age cohorts are affected differentially by exogenous shocks. However, as the dimensions and degree of heterogeneity in these models increase, the computational burden imposed by rational expectations solution methods for non-stationary equilibrium time paths increases exponentially. As a result, these models have been limited in the scope of their use to a restricted set of applications and a relatively small group of researchers. This paper presents two alternative methods for solving for non-stationary equilibrium time paths in OLG life cycle models that are new to this class of model. We then compare the speed and accuracy of these computational methods with the benchmark rational expectations time-path iteration method.

keywords: Computable General Equilibrium Models, Heterogeneous Agents, Overlapping Generations Model, Distribution of Savings

JEL classification: C63; C68; D31; D91

*We are grateful to the Brigham Young University *R² Seminar Series* participants for helpful comments.

[†]Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

[‡]Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, (801) 422-5928, kerk_phillips@byu.edu.

1 Introduction

In 2008, the overlapping generations (OLG) model proposed by Samuelson (1958) turned 50.¹ OLG models provide a dynamic general equilibrium setting with heterogeneous agents that looks more simple and intuitive on the surface than the more conventional models with infinitely lived agents. The OLG framework is invaluable for analyzing any type of question in which age cohorts are affected differentially by exogenous shocks. However, the intuitive structure of new generations of finitely lived agents being born in each period comes at a cost. The first two Fundamental Welfare Theorems do not hold in OLG models, and the computation of non-stationary equilibrium time paths of the endogenous variables can require a tremendous computational burden.

It is the latter complexity of OLG models—namely, the computational burden of computing non-stationary equilibrium time paths—that we wish to address in this paper. In particular, we propose two new methods for computing the non-stationary equilibrium time paths for endogenous variables. Neither method is new to economics, but rather has not been employed in the OLG class of dynamic general equilibrium models. As both methods employ some degree of approximation, we compare the two new solution methods to the benchmark conventional method in terms of both speed and accuracy.

As a recent example of the constraints exacted by the computational requirements for equilibrium transition paths in OLG models with a significant degree of heterogeneity and uncertainty, Nishiyama and Smetters (2007) lament that “[t]he more extensive model contained in this paper requires the addition of another state variable, which significantly increases the...required computation time from several hours to typically several days per simulation.” Even a computation time of several hours makes procedures like forecasting prohibitive if the model must be simulated numerous times in order to approximate the distribution of forecasts.

¹See Weil (2008) and Solow (2006). The authors do not know what was the earliest reference to the strange OLG acronym. However, Karl Shell still refers to it, in a manner more consistent with the rules of acronyms, as the OG model.

The benchmark conventional solution method for the non-stationary equilibrium time path in OLG models is outlined in [Auerbach and Kotlikoff \(1987, ch. 4\)](#) for the perfect foresight case and in [Nishiyama and Smetters \(2007, Appendix II\)](#) for the stochastic case. We call this method transition path iteration (TPI). We will show the conventional TPI method is a rational expectations equilibrium concept in which each agent correctly forecasts the decisions of the other agents, thereby correctly forecasting the future distribution of savings.

The first new OLG solution method we propose is the method of undetermined coefficients (MUC) from [Christiano \(2002\)](#). This method has been used widely in computing solutions to dynamic stochastic general equilibrium (DSGE) models with infinitely lived agents. MUC simplifies the equilibrium computation by taking a linear approximation of the characterizing intertemporal Euler equations. We call our second new OLG solution method the alternative model forecast (AMF) method. This method approximates the rational expectations requirement by having agents forecast future states of the world using some general alternative model method. This approximation can reduce the equilibrium time path computation time if the degree of heterogeneity in the OLG model is large enough. It is essentially a weakening of the rational expectations assumption.

We find that no solution method computes the equilibrium time path faster than the method of undetermined coefficients (MUC). The speed of this computation comes from the linear approximation of the characterizing intertemporal Euler equations which eliminates the need for recursive solution techniques. We also find the accuracy lost in the MUC solution path is minimal. In addition, we find that the alternative model forecast (AMF) method can perform faster than the benchmark TPI solution method if the degree of heterogeneity in the OLG model is large enough. The accuracy of the AMF method depends on the alternative model used.

This paper is organized as follows. [Section 2](#) describes a simple perfect-foresight version of the model with S -period lived agents and defines the steady-state equilibrium and the transition path equilibrium. [Section 3](#) describes the benchmark equilibrium time path solution method and the two new solution methods, and compares

each method in terms of speed and accuracy. Section 4 then compares the three solution methods' speed and accuracy on a large stochastic version of the model similar to Nishiyama and Smetters (2007) which incorporates a larger degree of heterogeneity. Section 5 concludes.

2 General perfect foresight model: S -period lived agents

2.1 Household problem

A measure $1/S$ of identical individuals is born in each period t and live for $S \in (1, \infty)$ periods. Individuals are endowed with one unit of time in each period t that they supply inelastically to the labor market. Let s represent the periods that an individual has been alive. The fixed labor supply in each period t by each age- s individual is denoted by $n_{s,t}$.

At time t , generation s agents know the real wage rate w_t and know the one-period real interest rate r_t on bonds $b_{s,t}$ that mature at the beginning of period t . In each period t , age- s agents choose how much to consume $c_{s,t}$ and how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{s+1,t+1}$ in order to maximize expected lifetime utility of the following form,

$$U_{s,t} = \sum_{u=0}^{S-s} \beta^u u(c_{s+u,t+u}) \quad \text{where} \quad u(c_{s,t}) = \frac{(c_{s,t})^{1-\sigma} - 1}{1-\sigma} \quad (2.1)$$

where $u(c)$ is a constant relative risk aversion utility function, $\sigma > 0$ is the coefficient of relative risk aversion, and $\beta \in (0, 1)$ is the agent's discount factor

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life S , the per-period budget constraints for each agent

normalized by the price of consumption are the following.

$$w_t n_{s,t} \geq c_{s,t} + b_{s+1,t+1} \quad \text{for } s = 1 \quad (2.2)$$

$$(1 + r_t) b_{s,t} + w_t n_{s,t} \geq c_{s,t} + b_{s+1,t+1} \quad \text{for } 2 \leq s \leq S - 1 \quad (2.3)$$

$$(1 + r_t) b_{s,t} + w_t n_{s,t} \geq c_{s,t} \quad \text{for } s = S \quad (2.4)$$

Adding up these per-period budget constraints gives the following representations of each agent's lifetime budget constraint.²

$$\sum_{u=0}^{S-1} d_{t+u} w_{t+u} n_{s+u,t+u} \geq \sum_{u=0}^{S-1} d_{t+u} c_{s+u,t+u} \quad \text{for } s = 1 \quad (2.5)$$

$$(1 + r_t) b_{s,t} + \sum_{u=0}^{S-s} d_{t+u} w_{t+u} n_{s+u,t+u} \geq \sum_{u=0}^{S-s} d_{t+u} c_{s+u,t+u} \quad \text{for } 2 \leq s \leq S \quad (2.6)$$

where

$$d_{t+u} = \begin{cases} 1 & \text{if } u = 0 \\ \prod_{v=1}^u \frac{1}{1+r_{t+v}} & \text{if } 1 \leq u \leq S \end{cases} \quad (2.7)$$

We next describe the Euler equations that govern the choices of consumption $c_{s,t}$ and savings $b_{s+1,t+1}$ by household of age s in each period t . We work backward from the last period of life $s = S$. Because households do not save in the last period of life $b_{s+1,t+1} = 0$ due to our assumption of no bequest motive, the household's final-period maximization problem is given by the following.

$$\max_{c_{S,t}} \frac{(c_{S,t})^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.} \quad (1 + r_t) b_{S,t} + w_t n_{S,t} \geq c_{S,t} \quad \forall t \quad (2.8)$$

Because the $s = S$ problem (2.8) involves only time- t variables that are known with certainty, the solution to the problem is trivially that the household consumes all of its income in the last period of life.

$$c_{S,t} = (1 + r_t) b_{S,t} + w_t n_{S,t} \quad \forall t \quad (2.9)$$

²See Appendix T-1 for derivation of the lifetime budget constraint.

In general, maximizing (2.1) with respect to either (2.5) or (2.6) gives the following intratemporal Euler equation,

$$\beta^u (c_{s+u,t+u})^{-\sigma} = \lambda d_{t+u} \quad \text{for } 0 \leq u \leq S - s, 1 \leq s \leq S, \forall t \quad (2.10)$$

where λ is the Lagrangian multiplier on the lifetime budget constraint. The intertemporal Euler equation is obtained by iterating (2.10) forward one period to $t + 1$, and dividing it by the period- t intratemporal Euler equation.

$$\begin{aligned} (c_{s+u,t+u})^{-\sigma} &= \beta (1 + r_{t+u+1}) (c_{s+u+1,t+u+1})^{-\sigma} \\ \text{for } 1 \leq s \leq S - 1, \quad 0 \leq u \leq S - s, \quad \forall t \end{aligned} \quad (2.11)$$

2.2 Firm problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = AK_t^\alpha N_t^{1-\alpha} \quad \forall t \quad (2.12)$$

where A is the fixed technology process and $\alpha \in (0, 1)$. Profit maximization results in the real wage w_t and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively.

$$w_t = (1 - \alpha) \frac{Y_t}{N_t} \quad \forall t \quad (2.13)$$

$$r_t = \alpha \frac{Y_t}{K_t} \quad \forall t \quad (2.14)$$

2.3 Market clearing and equilibrium

Labor market clearing requires that aggregate labor demand N_t equal the sum of individual labor supplied $n_{s,t}$. The supply side of market clearing in the labor market is trivial because household labor is supplied inelastically. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by

households $b_{s,t}$. Lastly, goods market clearing requires that aggregate production Y_t equal the sum of household consumption $c_{s,t}$.

$$N_t = \sum_{s=1}^S \frac{1}{S} n_s \quad \forall t \quad (2.15)$$

$$K_t = \sum_{s=2}^S \frac{1}{S} b_{s,t} \quad \forall t \quad (2.16)$$

$$Y_t = \sum_{s=1}^S \frac{1}{S} c_{s,t} \quad \forall t \quad (2.17)$$

Let $\mathbf{b}_t \equiv \{b_{2,t}, b_{3,t}, \dots, b_{S,t}\}$ be the distribution of capital at time t . Then the steady-state equilibrium for this economy is defined as follows.

Definition 1 (Steady-state equilibrium in simple S -period model). A non-autarkic steady-state equilibrium in the simple overlapping generations model with S -period lived agents is defined as a constant distribution of capital

$$\mathbf{b}_t = \{\bar{b}_2, \bar{b}_3, \dots, \bar{b}_S\} = \bar{\mathbf{b}} \quad \forall t$$

consumption allocations $c_{s,t}$ for all s and t , aggregate firm production Y_t , labor demand H_t , and prices w_t and r_t for all t such that households optimize according to (2.11), firms optimize according to (2.13) and (2.14), and markets clear according to (2.15), (2.16), and (2.17).

The steady-state equilibrium of the simple S -period OLG model is relatively simple to solve because the steady-state distribution assumption that $\mathbf{b}_t = \bar{\mathbf{b}}$ implies the following:

$$\mathbf{b}_t = \bar{\mathbf{b}} \quad \Rightarrow \quad b_{s,t} = b_{s,t+u} = \bar{b}_s \quad \forall s, t, \quad \text{and} \quad 1 - s \leq u \leq S - s \quad (2.18)$$

This relieves the burden of knowing what all the other agents in the economy are doing.

From (2.16), it is clear that the aggregate capital stock K_t is a function of the state \mathbf{b}_t . The only nontrivial variable in the production function (2.12) is K_t , so aggregate output Y_t is also just a function of the state \mathbf{b}_t . The first order conditions

of the firm (2.13) and (2.14) show that prices w_t and r_t are functions of Y_t and K_t and, therefore, functions of the state $r_t(\mathbf{b}_t)$ and $w_t(\mathbf{b}_t)$. Expression (2.18) suggests that the distribution of capital in the steady state is equivalent to the distribution of capital across any individual household's lifespan.

$$\mathbf{b}_t = \bar{\mathbf{b}} = \{\bar{b}_2, \bar{b}_3, \dots, \bar{b}_S\} \quad \Rightarrow \quad \{\bar{b}_{2,t}, \bar{b}_{3,t}, \dots, \bar{b}_{S,t}\} = \{\bar{b}_{2,t}, \bar{b}_{3,t+1}, \dots, \bar{b}_{S,t+S-1}\} \quad (2.19)$$

Expression (2.19) implies that the $S - 1$ savings rates over the life span of a household in the steady state are perfectly identified by the system of $S - 1$ equations represented by the household's intertemporal Euler equations (2.11). Substituting the steady-state version of the per-period budget constraint (2.3) into the intertemporal Euler equation (2.11) gives the following steady-state system of $S - 1$ equations in $S - 1$ unknowns.

$$\begin{aligned} & \left[\left(1 + \bar{r}(\bar{\mathbf{b}})\right) \bar{b}_s + \bar{w}(\bar{\mathbf{b}}) n_s - \bar{b}_{s+1} \right]^{-\sigma} = \dots \\ & \beta \left(1 + \bar{r}(\bar{\mathbf{b}})\right) \left[\left(1 + \bar{r}(\bar{\mathbf{b}})\right) \bar{b}_{s+1} + \bar{w}(\bar{\mathbf{b}}) n_{s+1} - \bar{b}_{s+2} \right]^{-\sigma} \quad (2.20) \\ & \text{for } 1 \leq s \leq S - 1, \quad \text{and } \bar{b}_1 = \bar{b}_{S+1} = 0 \end{aligned}$$

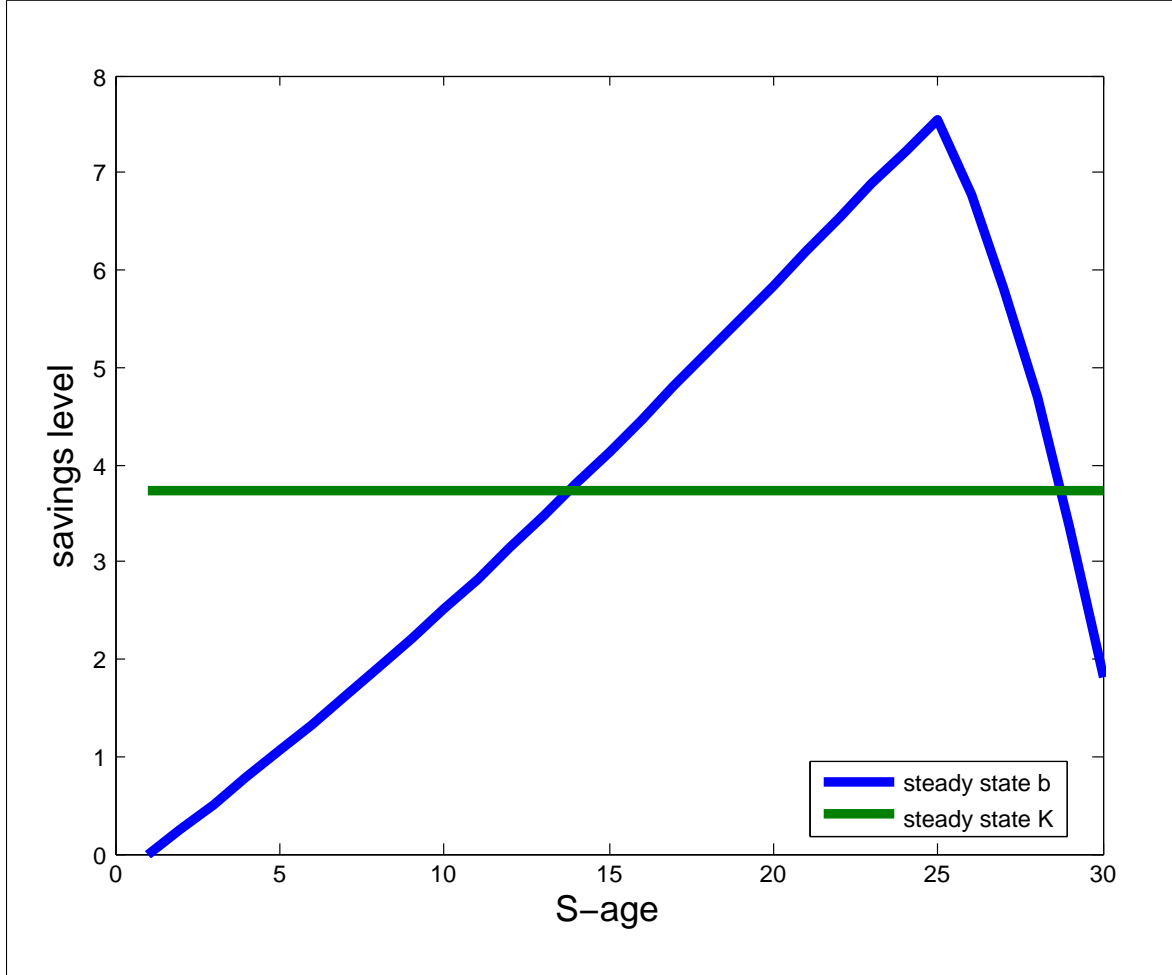
Because the an analytical solution to (2.20) is not possible due to the CRRA preferences, the steady-state equilibrium distribution of savings $\bar{\mathbf{b}}$ must be solved numerically.

Figure 1 shows the computed steady-state equilibrium distribution of savings $\bar{\mathbf{b}}$ over the life cycle for a particular calibration of the model parameters $[S, \beta, \sigma, \alpha] = [30, 0.96, 2, 0.35]$. We assume that labor is supplied inelastically for the first 24 periods of life ($n_s = 1$ for $1 \leq s \leq 24$) and that the labor supply goes to zero for the last six periods ($n_s = 0$ for $s \geq 25$), which corresponds to a retirement period of the last 20-percent of household's lifespan.³ The steady-state aggregate capital stock shown

³A person's lifespan here is defined as the duration from the period they start working until the period they die. We ignore childhood. So the calibration of $n_s = 1$ for $1 \leq s \leq 24$ and $n_s = 0$ for $25 \leq s \leq 30$ corresponds to households starting work at age 20, working until age 65, being in retirement from age 66 to age 75, and then dying at the end of 75.

in Figure 1 is $\bar{K}(\bar{\mathbf{b}}) = 3.74$, and the steady-state equilibrium real wage and real interest rate are $\bar{w}(\bar{\mathbf{b}}) = 1.12$ and $\bar{r}(\bar{\mathbf{b}}) = 0.13$, respectively.

Figure 1: Steady-state distribution of savings $\bar{\mathbf{b}}$: $S = 30$



Outside of the steady state, an age- s household's intertemporal consumption decision in each period from (2.11) depends not only on the current state \mathbf{b}_t and that household's savings decision in the current period $b_{s+1,t+1}$ and the future period $b_{s+2,t+2}$, but also on the savings decisions of all the other households in the economy

in the current period \mathbf{b}_{t+1} .

$$\begin{aligned} & \left[\left(1 + r(\mathbf{b}_t) \right) b_{s,t} + w(\mathbf{b}_t) n_s - b_{s+1,t+1} \right]^{-\sigma} = \dots \\ \beta \left(1 + r(\mathbf{b}_{t+1}) \right) & \left[\left(1 + r(\mathbf{b}_{t+1}) \right) b_{s+1,t+1} + w(\mathbf{b}_{t+1}) n_{s+1} - b_{s+2,t+2} \right]^{-\sigma} \quad (2.21) \\ & \text{for } 1 \leq s \leq S-1, \quad \text{and } b_{1,t} = b_{S+1,t} = 0 \quad \forall t \end{aligned}$$

The non-steady-state equilibrium in this economy is more complicated because relation (2.19) does not hold when the economy is not in the steady state. In this case, the real wage and the real interest rate in each period are functions of the distribution of savings at the beginning of that period $w_t = w(\mathbf{b}_t)$ and $r_t = r(\mathbf{b}_t)$. This means that each household must be able to forecast future prices, and therefore future capital distributions, in order to make its own savings decisions. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the linear operator $\Gamma(\cdot)$ such that:

$$\mathbf{b}_{t+u}^e = \Gamma^u(\mathbf{b}_t) \quad \forall t, \quad u \geq 1 \quad (2.22)$$

where the e superscript signifies that \mathbf{b}_{t+u}^e are general beliefs that are not constrained to be correct.

The expression of individual beliefs in (2.22) is a weak assumption in the sense that it does not constrain the beliefs to be correct. However, it is a strong assumption in that it implies the following two properties. First (2.22) implies that each household knows the entire distribution of savings \mathbf{b}_t at time t . It also implies that each household has symmetric beliefs about the savings policy function of all the other households $b_{s+1,t+1} = b_{s+1}(\mathbf{b}_t)$. That is, $\Gamma(\cdot)$ has no s subscript. With these general beliefs, we can now define a general non-steady-state equilibrium of the simple S -period model.

Definition 2 (Non-steady-state equilibrium in simple S -period model). A non-steady state equilibrium in the simple overlapping generations model with S -

period lived agents is defined as a distribution of capital $b_{s,t}$ and consumption allocations $c_{s,t}$ for all s and t , aggregate firm production Y_t , labor demand H_t , and prices w_t and r_t for all t such that:

- i. households have symmetric beliefs about the future savings decisions of the other agents characterized by the transition function $\Gamma(\cdot)$ described in (2.22) such that they believe the transition of the state in each period evolves in the following way,

$$\mathbf{b}_{t+u}^e = \Gamma^u(\mathbf{b}_t) \quad \forall t, \quad u \geq 1$$

- ii. household beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{b}_{t+u}^e = \mathbf{b}_{t+u} \quad \forall t, \quad u \geq 1$$

- iii. households optimize according to (2.11),
 - iv. firms optimize according to (2.13) and (2.14),
 - v. and markets clear according to (2.15), (2.16), and (2.17).
-

One implication of households having symmetric beliefs is that they will have symmetric policy functions. In other words, (2.22) implies the following.

$$\mathbf{b}_{t+u}^e = \Gamma^u(\mathbf{b}_t) \quad \forall t, \quad u \geq 1 \quad \Rightarrow \quad b_{s,t+1}(\mathbf{b}_t) = b_{s,t+u}(\mathbf{b}_t) \quad \forall s, t, \quad u \neq 1 \quad (2.23)$$

That is, if the equilibrium savings choice is $b_{s,t+1}$ according to Definition 2 for an age- s household given the state \mathbf{b}_t , then an age- s household in a different period $t+u$ will choose the same equilibrium savings rate if the same state \mathbf{b}_t occurs. The intuition is that if a household knows what savings level it would choose at any age s and any state \mathbf{b}_t , then the symmetry of the problem implies that the household knows what all the other households would choose at any age s or state \mathbf{b}_t .

With Definition 2, the non-steady-state equilibrium can be computed by rewriting the set of intertemporal Euler conditions from (2.21) in the following way.

$$\left[\left(1 + r(\mathbf{b}_t)\right) b_{s,t} + w(\mathbf{b}_t) n_s - b_{s+1,t+1} \right]^{-\sigma} = \dots$$

$$\beta \left(1 + r(\Gamma(\mathbf{b}_t))\right) \left[\left(1 + r(\Gamma(\mathbf{b}_t))\right) b_{s+1,t+1} + w(\Gamma(\mathbf{b}_t)) n_{s+1} - b_{s+2,t+2} \right]^{-\sigma} \quad (2.24)$$

$$\text{for } 1 \leq s \leq S-1, \quad \text{and } b_{1,t} = b_{S+1,t} = 0 \quad \forall t$$

System (2.24) is then a set of $S - 1$ equations with $S - 1$ policy functions $b_s(\cdot)$. The fixed point argument of the non-steady state equilibrium is that the set of $S - 1$ policy functions shown in (2.23) implied by beliefs (2.22) must be consistent across time as described in part (ii) of Definition 2.

3 Non-steady-state Equilibrium Solution Methods

This section outlines the benchmark time path iteration (TPI) method for solving the non-steady-state equilibrium path of the distribution of savings and then details two new methods for computing the equilibrium transition path—the method of undetermined coefficients (MUC) and alternative model forecast (AMF) methods. As discussed in the introduction, the two new methods (MUC and AMF) are only new in their application to OLG models. They have been widely used in the DSGE literature with infinitely lived agents. Because the MUC and AMF solution methods represent approximations of some dimension of the benchmark TPI method, we compare the MUC and AMF methods to the benchmark TPI method in terms of both speed and accuracy.

3.1 Benchmark: Time-path iteration

The most common method of solving for non-steady-state equilibrium time path for the capital distribution in OLG models is finding a fixed point for the time-path of the distribution of capital for each possible initial state of the distribution of capital. This solution method is detailed for the perfect foresight case in Auerbach and Kotlikoff (1987, ch. 4) and for the stochastic case in Nishiyama and Smetters (2007, Appendix II). The idea is that the economy is infinitely lived, although the agents that make up the economy are not. So rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire time path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady state equilibrium $\bar{\mathbf{b}}$

in a finite number of periods $T < \infty$ regardless of the initial state \mathbf{b}_0 . The first step is to assume a time path for aggregate capital $K^0 = \{K_1^0, K_2^0, \dots, K_T^0\}$ such that T is sufficiently large to ensure that $\mathbf{b}_T = \bar{\mathbf{b}}$ and $K_T^0 = \bar{K}$. The superscript 0 is an index for the iteration number. This time path for aggregate capital determines the time path for both the real wage $w^0 = \{w_1^0, w_2^0, \dots, w_T^0\}$ and the real return on investment $r^0 = \{r_1^0, r_2^0, \dots, r_T^0\}$. The exact initial distribution of capital in the first period $b_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$ can be arbitrarily chosen as long as it satisfies $K_1^0 = \sum_{s=2}^S \frac{1}{s} b_{s,1}$ according to market clearing condition (2.16). One could also first choose the initial distribution of capital $b_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$ and then choose an initial aggregate capital stock K_1^0 that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial time path for aggregate capital is that it equal the steady state level $K_T^0 = \bar{K} = \sum_{s=2}^S \frac{1}{s} \bar{b}_s$ by period T . But the aggregate capital stocks for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\mathbf{b}_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$ and the time paths of aggregate capital $K^0 = \{K_1^0, K_2^0, \dots, K_T^0\}$, the real wage $w^0 = \{w_1^0, w_2^0, \dots, w_T^0\}$, and the real return to investment $r^0 = \{r_1^0, r_2^0, \dots, r_T^0\}$, one can solve for the optimal level of savings by the $S-1$ -aged agent for the last period of his life $b_{S,2}$ using his intertemporal Euler equation:

$$[(1 + r_1^0) b_{S-1,1} + w_1^0 n_{S-1} - b_{S,2}]^{-\sigma} = \beta (1 + r_2^0) [(1 + r_2^0) b_{S,2} + w_2^0 n_S]^{-\sigma} \quad (3.1)$$

Because r_1^0 , w_1^0 and $b_{S-1,1}$ were decided by the choice of initial state and because r_2^0 and w_2^0 were assumed, $b_{S,2}$ is the savings value that makes (3.1) hold.

The final two savings decisions of the $S-2$ -aged household in period 1, $b_{S-1,2}$ and $b_{S,3}$, are characterized by the following two intertemporal Euler equations.

$$\begin{aligned} [(1 + r_1^0) b_{S-2,1} + w_1^0 n_{S-2} - b_{S-1,2}]^{-\sigma} &= \beta (1 + r_2^0) [(1 + r_2^0) b_{S-1,2} + w_2^0 n_{S-1} - b_{S,3}]^{-\sigma} \\ [(1 + r_2^0) b_{S-1,2} + w_2^0 n_{S-1} - b_{S,3}]^{-\sigma} &= \beta (1 + r_3^0) [(1 + r_3^0) b_{S,3} + w_3^0 n_S]^{-\sigma} \end{aligned} \quad (3.2)$$

You continue solving for the lifetime savings decisions of each household alive in period

1 until you solve for the lifetime savings decisions of the one-period-old household in period 1. These $S - 1$ savings decisions are characterized by the following $S - 1$ equations.

$$\begin{aligned}
[w_1^0 n_1 - b_{2,2}]^{-\sigma} &= \beta (1 + r_2^0) [(1 + r_2^0) b_{2,2} + w_2^0 n_2 - b_{3,3}]^{-\sigma} \\
[(1 + r_2^0) b_{2,2} + w_2^0 n_2 - b_{3,3}]^{-\sigma} &= \beta (1 + r_3^0) [(1 + r_3^0) b_{3,3} + w_3^0 n_3 - b_{4,4}]^{-\sigma} \\
&\vdots \\
[(1 + r_{S-1}^0) b_{S-1,S-1} + w_{S-1}^0 n_{S-1} - b_{S,S}]^{-\sigma} &= \beta (1 + r_S^0) [(1 + r_S^0) b_{S,S} + w_S^0 n_S]^{-\sigma}
\end{aligned} \tag{3.3}$$

Once you have solved for all the remaining lifetime savings decisions of each household alive in period 1, you have solved for the complete set of savings decisions in the period-2 savings distribution $\mathbf{b}_2^0 = \{b_{2,2}, b_{3,2}, \dots, b_{S,2}\}$. Using this newly computed period-2 savings distribution, we can calculate a new aggregate capital stock for period 2 using (2.16) $K_2^{0'} = \sum_2^S \frac{1}{S} b_{s,2}$. This, in turn, generates new period-2 values for $w_2^{0'}$ and $r_2^{0'}$.

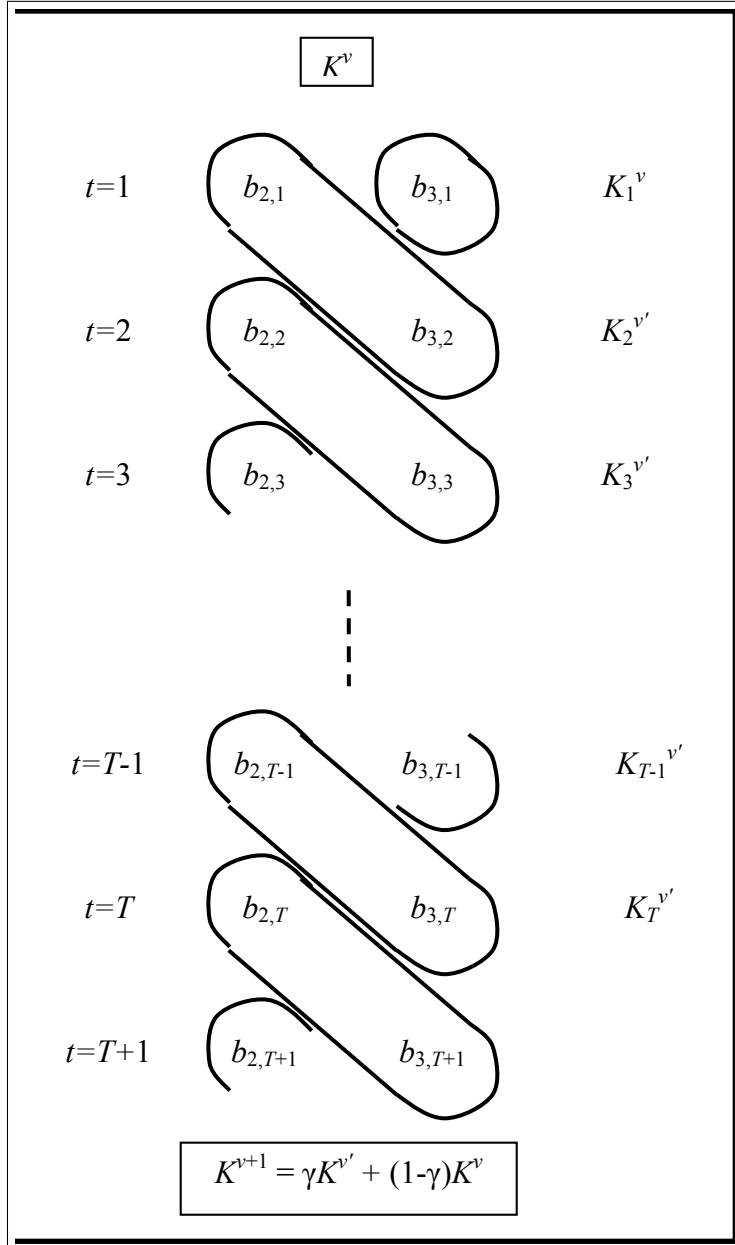
A new period-3 aggregate capital stock $K_3^{0'}$ is found in the same way by solving the system of $S - 1$ equations similar to (3.3) for $S - 1$ savings rates over the life cycle of the household born in period 2. This process, illustrated in Figure 2 for the $S = 3$ case, is repeated until new values have been generated for $K_{T-1}^{0'}$, $w_{T-1}^{0'}$, and $r_{T-1}^{0'}$. Recall that the time- T values of the time paths were constrained to equal their steady state values.

Once the entire new time path of aggregate capital $K^{0'}$ is calculated, a new time path is assumed that is a convex combination (a contraction) of the initially assumed time path K^0 and the newly calculated time path $K^{0'}$.

$$K^1 = \gamma K^{0'} + (1 - \gamma) K^0 \quad \forall \gamma \in (0, 1) \tag{3.4}$$

The process of calculating new values for the time path of aggregate capital $K^{1'}$ is repeated, and the convex combination of the initial time path K^1 and the newly calculated time path $K^{1'}$ form a new time path of capital K^2 . For any integer $u > 0$,

Figure 2: Time path iteration diagram: $S = 3$ example



the next guess for a time path of aggregate capital K^{u+1} is given by the following equation.

$$K^{u+1} = \gamma K^{u'} + (1 - \gamma)K^u \quad \forall u \geq 0, \gamma \in (0, 1) \quad (3.5)$$

The equilibrium time path of the distribution of capital $\{\mathbf{b}_t^*\}_{t=1}^T$ given the initial distribution of capital \mathbf{b}_1 and the steady state distribution of capital $\mathbf{b}_T = \bar{\mathbf{b}}$ is the time path of the distribution of capital that is consistent with the time path of the level of aggregate capital K^u . This equilibrium distribution of capital is found when the calculated time path of aggregate capital $K^{u'}$ is arbitrarily close to the assumed time path of aggregate capital K^u . Each distribution of capital \mathbf{b}_t^* and the corresponding time- t allocations and prices in the computed equilibrium time path $\{\mathbf{b}_t^*\}_{t=1}^T$ represents a non-steady-state equilibrium as defined in Definition 2.

Definition 3 (Time path iteration (TPI) equilibrium time path of the distribution of capital). Given some initial distribution of capital $\mathbf{b}_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$, a steady-state distribution of capital arrived at after T periods $\bar{\mathbf{b}}_T = \{\bar{b}_2, \bar{b}_3, \dots, \bar{b}_S\}$ the equilibrium time path of the distribution of capital $\{\mathbf{b}_t^*\}_{t=1}^T$ is defined as the distribution of capital for which the calculated level of aggregate capital $K^{u'}$ is arbitrarily close to the initial guess for the time path of aggregate capital K^u .

$$\{\mathbf{b}_t^*\}_{t=1}^T \equiv \{\mathbf{b}_t\}_{t=1}^T : |K^{u'} - K^u| < \varepsilon$$

where $|\cdot|$ is a general distance metric and ε is a positive number arbitrarily close to zero. Each distribution of capital \mathbf{b}_t^* in the equilibrium time path $\{\mathbf{b}_t^*\}_{t=1}^T$ represents a non-steady-state equilibrium distribution of capital according to Definition 2 for the given state \mathbf{b}_{t-1} in the previous period.

The equilibrium non-steady-state distributions of capital defined in Definition 3 imply the beliefs $\Gamma(\cdot)$ required by Definition 2 because $|K^{u'} - K^u| < \varepsilon$ implies that each household can correctly forecast the other household's decisions $\mathbf{b}_{t+u} = \Gamma^u(\mathbf{b}_t)$. The policy functions for the entire state space can be computed by solving for the equilibrium time paths $\{\mathbf{b}_t^*\}_{t=1}^T$ for all possible initial distributions of capital \mathbf{b}_1 .

3.2 Linearization: Method of undetermined coefficients

The method of undetermined coefficients (MUC) for solving dynamic general equilibrium models has been standardized by [Christiano \(2002\)](#). However, this method has not been employed in solving for non-stationary transition path equilibria in OLG models. MUC leverages the linearity of the policy functions that result from linearized Euler equations in order to simplify the computation required. In effect, equilibria solved by MUC require only the period- t Euler equations for all households alive in period- t in order to solve for the policy functions $\mathbf{b}_{t+1} = \Gamma(\mathbf{b}_t) = \mathbf{A}\mathbf{b}_t + \mathbf{C}$, whereas the benchmark TPI method from [Section 3.1](#) requires all the remaining Euler equations for each household alive in period t .

The general period- t intertemporal Euler equation for an age- s household rewritten in $g(x) = 0$ form is the following.

$$\begin{aligned} & [(1 + r_t) b_{s,t} + w_t n_s - b_{s+1,t+1}]^{-\sigma} - \dots \\ & \beta (1 + r_{t+1}) [(1 + r_{t+1}) b_{s+1,t+1} + w_{t+1} n_{s+1} - b_{s+2,t+2}]^{-\sigma} = 0 \end{aligned} \quad (3.6)$$

A first order Taylor series approximation of the intertemporal Euler equation around the steady state takes the form $g(x) \approx g(\bar{x}) + g'(\bar{x})\bar{x}\hat{x}$ where $\hat{x} = (x - \bar{x})/\bar{x}$. Applied to [\(3.6\)](#), the first order Taylor series approximation around the steady state is:

$$\begin{aligned} & (\bar{c}_s)^{-\sigma} - \beta (1 + \bar{r}) (\bar{c}_{s+1})^{-\sigma} - \sigma (1 + \bar{r}) (\bar{c}_s)^{-\sigma-1} \bar{b}_s \hat{b}_{s,t} + \dots \\ & [\sigma (\bar{c}_s)^{-\sigma-1} + \sigma \beta (1 + \bar{r})^2 (\bar{c}_{s+1})^{-\sigma-1}] \bar{b}_{s+1} \hat{b}_{s+1,t+1} - \beta (1 + \bar{r}) (\bar{c}_{s+1})^{-\sigma-1} \bar{b}_{s+2} \hat{b}_{s+2,t+2} = 0 \end{aligned} \quad (3.7)$$

Equation [\(3.7\)](#) is a second order linear difference equation in $\hat{b}_{s,t}$, $\hat{b}_{s+1,t+1}$, and $\hat{b}_{s+2,t+2}$, where $\hat{b}_{s,t} = (b_{s,t} - \bar{b}_s) / \bar{b}_s$.

The linearized Euler equations of all households alive at period t are given by the

following $S - 1$ equations.

$$\begin{aligned}
& (\bar{c}_1)^{-\sigma} - \beta(1 + \bar{r})(\bar{c}_2)^{-\sigma} - \sigma(1 + \bar{r})(\bar{c}_1)^{-\sigma-1}(0) + \dots \\
& \quad [\sigma(\bar{c}_1)^{-\sigma-1} + \sigma\beta(1 + \bar{r})^2(\bar{c}_2)^{-\sigma-1}] \hat{b}_{2,t+1} - \beta(1 + \bar{r})(\bar{c}_2)^{-\sigma-1} \hat{b}_{3,t+2} = 0 \\
& (\bar{c}_2)^{-\sigma} - \beta(1 + \bar{r})(\bar{c}_3)^{-\sigma} - \sigma(1 + \bar{r})(\bar{c}_2)^{-\sigma-1} \hat{b}_{2,t} + \dots \\
& \quad [\sigma(\bar{c}_2)^{-\sigma-1} + \sigma\beta(1 + \bar{r})^2(\bar{c}_3)^{-\sigma-1}] \hat{b}_{3,t+1} - \beta(1 + \bar{r})(\bar{c}_3)^{-\sigma-1} \hat{b}_{4,t+2} = 0 \quad (3.8) \\
& \quad \vdots \\
& (\bar{c}_{S-1})^{-\sigma} - \beta(1 + \bar{r})(\bar{c}_S)^{-\sigma} - \sigma(1 + \bar{r})(\bar{c}_{S-1})^{-\sigma-1} \hat{b}_{S-1,t} + \dots \\
& \quad [\sigma(\bar{c}_{S-1})^{-\sigma-1} + \sigma\beta(1 + \bar{r})^2(\bar{c}_S)^{-\sigma-1}] \hat{b}_{S,t+1} - \beta(1 + \bar{r})(\bar{c}_S)^{-\sigma-1}(0) = 0
\end{aligned}$$

The system of equations (3.8) can be rewritten in matrix notation as

$$\mathbf{X}\hat{\mathbf{b}}_t + \mathbf{Y}\hat{\mathbf{b}}_{t+1} + \mathbf{Z}\hat{\mathbf{b}}_{t+2} = \mathbf{K} \quad (3.9)$$

where \mathbf{X} , \mathbf{Y} , and \mathbf{Z} , are $(S - 1) \times (S - 1)$ dimension matrices, and $\hat{\mathbf{b}}_t$, $\hat{\mathbf{b}}_{t+1}$, $\hat{\mathbf{b}}_{t+2}$ and \mathbf{K} are $(S - 1) \times 1$ dimension vectors.

The method of undetermined coefficients assumes that the policy function for savings in each period is a linear function of the state

$$\hat{\mathbf{b}}_{t+1} = \mathbf{A}\hat{\mathbf{b}}_t + \mathbf{C} \quad \forall t \quad (3.10)$$

where \mathbf{A} is a constant $(S - 1) \times (S - 1)$ dimension matrix and \mathbf{C} is a constant $(S - 1) \times 1$ dimension vector. Substituting (3.10) into (3.9) for $\hat{\mathbf{b}}_{t+2}$ reduced (3.9) to a system of $S - 1$ first order linear difference equations. Solving those equations for $\hat{\mathbf{b}}_{t+1}$ as a function of $\hat{\mathbf{b}}_t$ gives the following form to the linear system of $S - 1$ policy functions.

$$\hat{\mathbf{b}}_{t+1} = (\mathbf{Y} + \mathbf{Z}\mathbf{A})^{-1}(-\mathbf{X})\hat{\mathbf{b}}_t + (\mathbf{Y} + \mathbf{Z}\mathbf{A})^{-1}(\mathbf{K} - \mathbf{Z}\mathbf{C}) \quad \forall t \quad (3.11)$$

The assumed form of the linear policy function (3.10) and the coefficient on $\hat{\mathbf{b}}_t$ in

(3.11) imply the following expression for \mathbf{A} .

$$\mathbf{A} = (\mathbf{Y} + \mathbf{Z}\mathbf{A})^{-1}(-\mathbf{X}) \quad (3.12)$$

Because \mathbf{A} appears on both sides of (3.12) and is not separable on the right-hand-side, Christiano (2002, section 3.1) shows how to solve for \mathbf{A} decomposing the right-hand-side of (3.12) into its eigenvector-eigenvalue decomposition. Once the solution for \mathbf{A} is found, \mathbf{C} can be solved analytically.

$$\mathbf{C} = [\mathbf{I} + (\mathbf{Y} + \mathbf{Z}\mathbf{A})^{-1}\mathbf{Z}]^{-1}(\mathbf{Y} + \mathbf{Z}\mathbf{A})^{-1}\mathbf{K} \quad (3.13)$$

where \mathbf{I} is an $(S - 1) \times (S - 1)$ dimension identity matrix.

With solutions for \mathbf{A} and \mathbf{C} in (3.10), the method of undetermined coefficients allows one to compute the distribution of savings tomorrow \mathbf{b}_{t+1} simply from the state today \mathbf{b}_t using only the $S - 1$ period- t intertemporal Euler equations for the households alive at period- t . Contrast this with the benchmark TPI method from Section 3.1 which required solving the entire series of remaining intertemporal Euler equations for each individual alive at time t . Each distribution of capital \mathbf{b}_{t+1} calculated from (3.10) and the corresponding time- t allocations and prices in the computed equilibrium time path $\{\mathbf{b}_t\}_{t=1}^T$ represents a non-steady-state equilibrium as defined in Definition 2.

Definition 4 (Method of undetermined coefficients (MUC) equilibrium time path of the distribution of capital). Given some initial distribution of capital $\mathbf{b}_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$, the equilibrium time path of the distribution of capital $\{\mathbf{b}_t\}_{t=1}^T$ is defined as the distribution of capital resulting from the linear policy function

$$\hat{\mathbf{b}}_{t+1} = \mathbf{A}\hat{\mathbf{b}}_t + \mathbf{C} \quad \forall t$$

where $\hat{\mathbf{b}}_t \equiv (\mathbf{b}_t - \bar{\mathbf{b}})/\bar{\mathbf{b}}$ and \mathbf{A} and \mathbf{C} are calculated using the method of undetermined coefficients described in Section 3.2 and in Christiano (2002). Each distribution of capital \mathbf{b}_t in the equilibrium time path $\{\mathbf{b}_t\}_{t=1}^T$ represents a non-steady-state equilibrium distribution of capital according to Definition 2 for the given state \mathbf{b}_{t-1} at time t .

The major difference between the benchmark TPI method for computing the non-

steady-state equilibrium time path for the distribution of capital from Definition 3 and the MUC method from Definition 4 is that the equilibrium time path from the MUC method can be computed period by period with just one run through the sample time period to time T . Additionally, the computation of each period's distribution only requires using $S - 1$ intertemporal Euler equations. In contrast, each period's distribution in the TPI method requires $\sum_{s=1}^{S-1} (S - s)$ unknowns and intertemporal Euler equations, and the entire time path of T distributions must be repeatedly computed until the assumed aggregate capital time path has converged.

Although the MUC method for computing the non-steady-state equilibrium time path of the distribution of capital is must faster than the benchmark TPI method, it also comes at a cost. The linearization of the intertemporal Euler equations around their steady state in (3.7) is an imperfect approximation of the true nonlinear relationship expressed in the original Euler equations from (3.6).

3.3 Relax rational expectations: Alternate model forecast

Our final method of solving for the non-steady state equilibrium time paths in OLG life cycle models is to use alternate model forecasts (AMF). This method can be thought of as relaxing the rational expectations requirement from parts (i) and (ii) of Definition 2 that each agent knows the policy function of all other agents. Instead, we make a weaker assumption that agents us some general alternative model to forecast in each period the time path of the aggregate capital stock $\{K_u^f, w_u^f, r_u^f\}_{u=t}^S$ for the remaining periods of their lives. This forecasted series is then updated each period when the value of the capital stock next period is realized.

Let $\Omega(\cdot)$ represent the general form of the alternative model each agent uses to forecast the time paths of the aggregate capital stock K_{t+s}^f , real wage w_{t+s}^f , and real rental rate r_{t+s}^f which are functions of the distribution of capital \mathbf{b}_t at time t . Then let the forecast for the aggregate capital stock be generated by the following general alternative model:

$$K_{t+1}^f = \Omega(K_t) \quad \Rightarrow \quad K_{t+s}^f = \Omega^s(K_t) \quad \text{s.t.} \quad \lim_{s \rightarrow \infty} \Omega^s(K_t) = \bar{K} \quad (3.14)$$

where w_{t+s}^f and r_{t+s}^f are just functions of K_{t+s}^f . The only condition that must be imposed on the alternative model is that the forecasts must go to the steady state in the limit $\lim_{s \rightarrow \infty} \Omega^s(K_t) = \bar{K}$. With $\{K_u^f, w_u^f, r_u^f\}_{u=t}^S$, each agent can choose their savings for the next period b_{t+1} as well as planned savings levels b_{t+s}^p for the remaining periods of life given the forecasted time path of the aggregate variables in the same way as described in equations (3.1) through (3.3) in Section 3.1.

At the end of period t , the distribution of capital for the next period \mathbf{b}_{t+1} has been decided and implies an aggregate capital stock that is not equal to the forecasted capital stock $K_{t+1}(\mathbf{b}_{t+1}) \neq K_{t+1}^f$, in general. With the new aggregate capital stock K_{t+1} , each agent repeats the process of forecasting the future values of the aggregate variables using the alternate model $\Omega(\cdot)$ until the time path reaches the steady state in period T . Each distribution of capital \mathbf{b}_{t+s} calculated using the alternative model method from (3.14) and the corresponding time- t allocations and prices in the computed equilibrium time path $\{\mathbf{b}_t\}_{t=1}^T$ represents an alternative model forecast non-steady-state equilibrium.

Definition 5 (Alternative model forecast (AMF) equilibrium time path of the distribution of capital). Given some initial distribution of capital $\mathbf{b}_1 = \{b_{2,1}, b_{3,1}, \dots, b_{S,1}\}$ and a steady-state distribution of capital arrived at after T periods $\bar{\mathbf{b}}_T = \{\bar{b}_2, \bar{b}_3, \dots, \bar{b}_S\}$, the alternative model forecast AMF equilibrium time path of the distribution of capital $\{\mathbf{b}_t^*\}_{t=1}^T$ is defined as the individual distributions of capital \mathbf{b}_t calculated by forecasting future aggregate variables using the alternative forecasting model $\Omega(\cdot)$ specified in equation (3.14) that is specified as follows:

$$K_{t+1}^f = \Omega(K_t) \quad \Rightarrow \quad K_{t+s}^f = \Omega^s(K_t) \quad \text{s.t.} \quad \lim_{s \rightarrow \infty} \Omega^s(K_t) = \bar{K}$$

Each individual distribution of capital is calculated using the remaining life forecasts of aggregate variables K_{t+s}^f according to equations (3.1) through (3.3).

3.4 Simple version comparison of solution methods

Given that each alternative method (MUC and AMF) is an approximation and simplification of the benchmark TPI method along some dimension, the goal of this

paper is to answer the question of how fast each alternative method is in computing the non-steady state equilibrium and how close each alternative method comes to the benchmark TPI method in accuracy. The benchmark time path iteration (TPI) method from Section 3.1 for computing the non-steady state equilibrium time path of the distribution of capital is the exact rational expectations equilibrium concept. Our two alternative methods for computing non-steady state equilibrium time paths approximate the benchmark method along some dimension. The method of undetermined coefficient (MUC) method from Section 3.2 takes a linear approximation of the intertemporal Euler equations (3.7) around the steady state. This greatly simplifies the solution method for the linear policy functions. In contrast, the alternative model forecast (AMF) method from Section 3.3 approximates the benchmark TPI method by using the alternative model $\Omega(\cdot)$ to forecast future aggregate variables rather than the TPI method's rational expectations law of motion operator $\Gamma(\cdot)$.

In this section, we calibrate the simple model we developed in Section 2 and compute the non-steady state time paths of the distribution of capital using the three methods described in Sections 3.1, 3.2, and 3.3. The number of periods an individual lives is $S = 30$, which corresponds to a period being equal to about 1.8 years if an individual's working life begins at age 20 and then dies at age 75 on average. We assume that individuals work for 80 percent of their lives (24 periods) and are retired for the final 20 percent of their lives on average. We normalize individual's inelastic labor supply in each period before retirement equal to $n_{s,t} = 1$ for $1 \leq s \leq 24$ and $n_{s,t} = 0$ for $s \geq 25$. In the utility function, we set the discount factor to $\beta = 0.94$, which is approximately a discount factor of 0.96 per year. We set a coefficient of relative risk aversion of $\sigma = 2$. In the firm production function, we set the Cobb-Douglas factor share of capital $\alpha = 0.35$ and we set the productivity parameter $A = 1$.

Lastly, for the alternative model forecast (AMF) method, we chose to use a naïve model for $\Omega(\cdot)$ in order to demonstrate a lower bound for proximity to our benchmark TPI time path. We assumed that agents forecast the next period's real wage rate w_{t+1} and real return on capital r_{t+1} to be a convex combination of the current period's

value and the steady state value:

$$\begin{aligned} w_{t+1}^f &= \exp\left(\rho \log(w_t) + (1 - \rho) \log(\bar{w})\right) \\ r_{t+1}^f &= \exp\left(\rho \log(r_t) + (1 - \rho) \log(\bar{r})\right) \end{aligned} \tag{3.15}$$

where $\rho = 0.43$. Note that this naïve forecasting model satisfies the required condition that $\lim_{s \rightarrow \infty} \Omega^s(x_t) = \bar{x}$ for the generic variable x_t .

Using these calibrated settings and the naïve AMF model, we computed the non-steady state equilibrium time paths for the distribution of capital using the TPI, MUC, and AMF methods from Sections 3.1, 3.2, and 3.3. Table 1 lists the computation times of the three methods, and Figure 3 shows the respective computed equilibrium time paths.

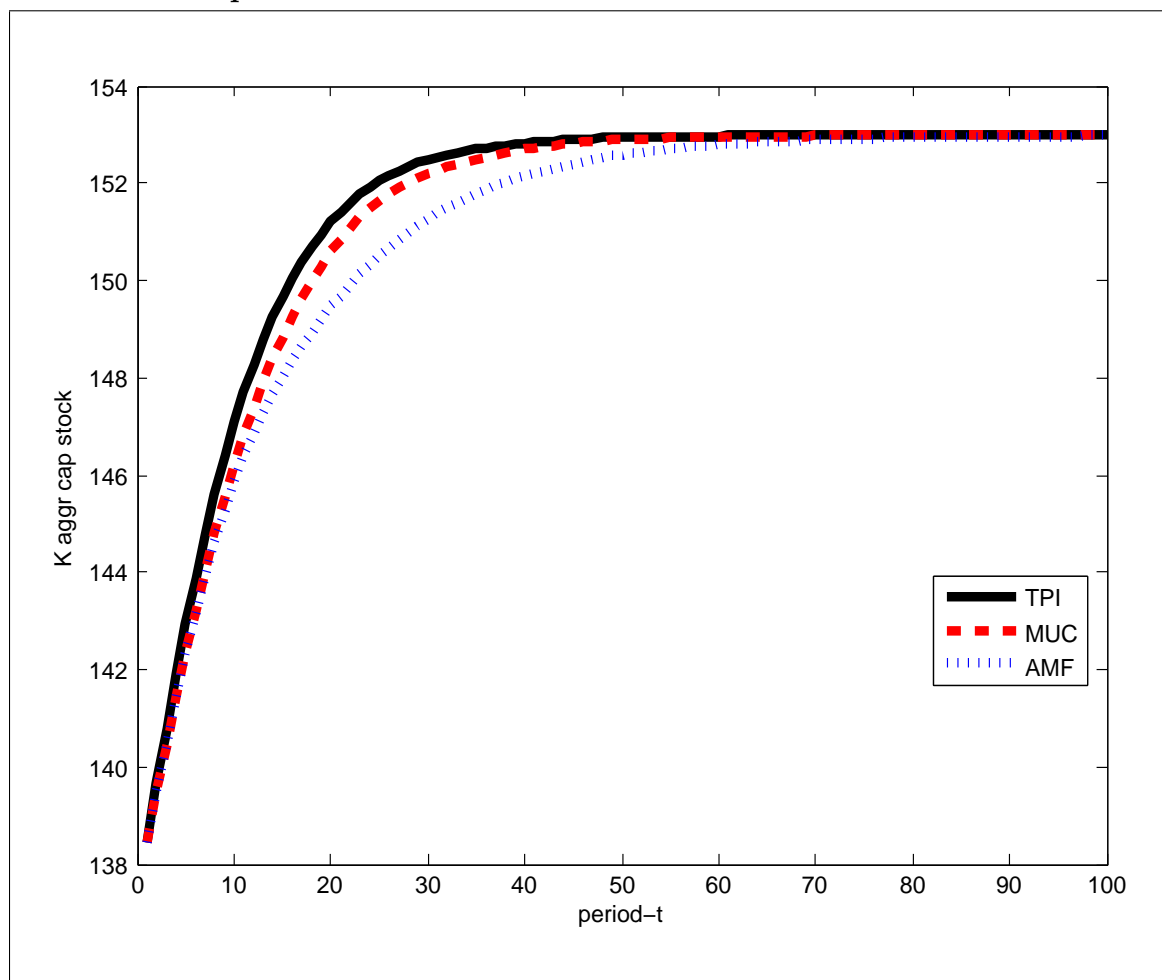
Table 1: Computation times for TPI, MUC, and AMF methods

Method	TPI	MUC	AMF
Speed (seconds)	6.2	1.0	15.1

In this simple model, the method of undetermined coefficients dominates the benchmark TPI and the alternative model forecast AMF computation methods in terms of speed. The linearization provided by MUC simplifies the computation in two main ways. First, the linear solution method is easier to compute than the non-linear solutions of TPI and AMF. However, the most important simplification is that the time path of aggregate capital does not have to be iteratively recalculated as in the TPI method or sequentially recalculated in the AMF method. Taken together, these two simplifications of the MUC method make it the fastest at calculating the non-steady state equilibrium time path for the distribution of capital.

Somewhat surprising is that the AMF method performed as poorly as it did along the computation speed dimension. The reason it was slower than the benchmark TPI method is that the TPI method converged quickly to the fixed point in the time path of aggregate capital. The AMF method had to calculate a new time path of aggregate

Figure 3: Time paths of aggregate capital stock K under TPI, MUC, AMF computation methods: $S = 30$



capital (real wage and real return) sequentially for the entire T periods. So if the iterations to convergence in the TPI method are much smaller than the number of periods T the computation must sequentially calculate in the AMF method, then the TPI method will be faster. This result is dependent on the simple model we used in this section to illustrate the solution methods. However, in Section 4 we will show that the AMF method becomes more efficient when the degree of heterogeneity and uncertainty are increased in the model.

As important as the speed required for the different solution methods to calculate the time paths is the accuracy of each method measured as deviation from the benchmark rational expectations TPI solution. Figure 3 shows the computed non-steady

state equilibrium time paths from the benchmark TPI, MUC, and AMF methods. An encouraging result is that the MUC method comes so close to the benchmark TPI method in this simple model. Also of interest is how close our AMF method comes to the TPI method given the naïveté of our forecast model from (3.15). Some type of more highly specified alternative model would likely move the AMF-computed time path closer to the benchmark TPI time path.

4 Comparison Using Big Stochastic Model

We are currently working on performing the same comparison among the TPI, MUC, and AMF computation methods as in Section 3.4 on a bigger model that has more heterogeneity and uncertainty. We are currently trying to do this with the model in [Nishiyama and Smetters \(2007\)](#).

5 Conclusion

APPENDIX

A-1 Proofs

Proof of Proposition 1: Existence and Uniqueness of Steady State. See [Stokey and Lucas \(1989, ch. 17\)](#) and [Li and Lin \(2008\)](#). \square

References

- AUERBACH, A. J., AND L. J. KOTLIKOFF (1987): *Dynamic Fiscal Policy*. Cambridge University Press.
- CHRISTIANO, L. J. (2002): “Solving Dynamic General Equilibrium Models by a Method of Undetermined Coefficients,” *Computational Economics*, 20(1-2), 21–55.
- GUSTMAN, A. L., AND T. STEINMEIER (2008): “How Changes in Social Security Affect Recent Retirement Trends,” NBER Working Paper 14105, National Bureau of Economic Research.
- KRUSELL, P., AND A. A. SMITH, JR. (1998): “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 106(5), 867–896.
- LI, J., AND S. LIN (2008): “Existence and Uniqueness of Steady-state Equilibrium in a Two-sector Overlapping Generations Model,” *Journal of Economic Theory*, 141(1), 255–275.
- NISHIYAMA, S., AND K. SMETTERS (2007): “Does Social Security Privatization Produce Efficiency Gains?,” *Quarterly Journal of Economics*, 122(4), 1677–1719.
- SAMUELSON, P. A. (1958): “An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money,” *Journal of Political Economy*, 66(6), 467–482.
- SOLOW, R. M. (2006): “Overlapping Generations,” in *Samuelsonian Economics and the Twenty-First Century*, ed. by M. Szenberg, L. Ramrattan, and A. A. Gottesman, pp. 34–41. Oxford University Press.
- STOKEY, N. L., AND R. E. LUCAS, JR. (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.
- WEIL, P. (2008): “Overlapping Generations: The First Jubilee,” *Journal of Economic Perspectives*, 22(4), 115–134.

TECHNICAL APPENDIX

T-1 Derivation of the Lifetime Budget Constraint

The lifetime budget constraint in year of life $s = S$ is simply the per-period budget constraint in (2.4).

$$(1 + r_t) b_{s,t} + w_t n_{s,t} \geq c_{s,t} \quad \text{for } s = S \quad (\text{T.1.1})$$

Iterating one period backward, the lifetime budget constraint for year of life $s = S - 1$ is characterized by the following two equations.

$$(1 + r_t) b_{s,t} + w_t n_{s,t} \geq c_{s,t} + b_{s+1,t+1} \quad \text{for } s = S - 1 \quad (\text{T.1.2})$$

$$(1 + r_{t+1}) b_{s+1,t+1} + w_{t+1} n_{s+1,t+1} \geq c_{s+1,t+1} \quad \text{for } s = S \quad (\text{T.1.3})$$

Solving (T.1.3) for $b_{s+1,t+1}$ and substituting the expression into (T.1.2) gives the following lifetime budget constraint for year of life $s = S - 1$.

$$(1 + r_t) b_{s,t} + w_t n_{s,t} + \left(\frac{1}{1 + r_{t+1}} \right) w_{t+1} n_{s+1,t+1} \geq \dots \quad (\text{T.1.4})$$

$$c_{s,t} + \left(\frac{1}{1 + r_{t+1}} \right) c_{s+1,t+1} \quad \text{for } s = S - 1$$

Iterating one more period backward, the lifetime budget constraint for year of life $s = S - 2$ is characterized by the following two equations,

$$(1 + r_t) b_{s,t} + w_t n_{s,t} \geq c_{s,t} + b_{s+1,t+1} \quad \text{for } s = S - 2 \quad (\text{T.1.5})$$

$$(1 + r_{t+1}) b_{s+1,t+1} + w_{t+1} n_{s+1,t+1} + \left(\frac{1}{1 + r_{t+2}} \right) w_{t+2} n_{s+2,t+2} \geq \dots \quad (\text{T.1.6})$$

$$c_{s+1,t+1} + \left(\frac{1}{1 + r_{t+2}} \right) c_{s+2,t+2} \quad \text{for } s = S - 1$$

where (T.1.6) is the lifetime budget constraint for year of life $s = S - 1$ when that year occurs in period $t + 1$. Solving (T.1.6) for $b_{s+1,t+1}$ and substituting the expression into (T.1.5) gives the following lifetime budget constraint for year of life $s = S - 2$.

$$(1 + r_t) b_{s,t} + w_t n_{s,t} + \left(\frac{1}{1 + r_{t+1}} \right) w_{t+1} n_{s+1,t+1} + \dots \quad (\text{T.1.7})$$

$$\left(\frac{1}{1 + r_{t+1}} \right) \left(\frac{1}{1 + r_{t+2}} \right) w_{t+2} n_{s+2,t+2} \geq \dots$$

$$c_{s,t} + \left(\frac{1}{1 + r_{t+1}} \right) c_{s+1,t+1} + \dots$$

$$\left(\frac{1}{1 + r_{t+1}} \right) \left(\frac{1}{1 + r_{t+2}} \right) c_{s+2,t+2} \quad \text{for } s = S - 2$$

By induction, the lifetime budget constraints for the years of life $2 \leq s \leq S$ and $s = 1$, respectively, are given by the following equations.

$$(1 + r_t) b_{s,t} + \sum_{u=0}^{S-s} d_{t+u} w_{t+u} n_{s+u,t+u} \geq \sum_{u=0}^{S-s} d_{t+u} c_{s+u,t+u} \quad \text{for } 2 \leq s \leq S \quad (\text{T.1.8})$$

$$\sum_{u=0}^{S-1} d_{t+u} w_{t+u} n_{s+u,t+u} \geq \sum_{u=0}^{S-1} d_{t+u} c_{s+u,t+u} \quad \text{for } s = 1 \quad (\text{T.1.9})$$

where

$$d_{t+u} = \begin{cases} 1 & \text{if } u = 0 \\ \prod_{v=1}^u \frac{1}{1+r_{t+v}} & \text{if } 1 \leq u \leq S \end{cases} \quad (\text{T.1.10})$$

T-2 Example: Two-period lived agents $S = 2$

In the two-period lived agent case $S = 2$, consumption in the last period of life $s = 2$ is given in (2.9) and represented below.

$$c_{2,t} = (1 + r_t) b_{2,t} + w_t n_{2,t} \quad \forall t \quad (\text{T.2.1})$$

When $S = 2$, households only have one savings decision $b_{2,t+1}$ at the end of the first period of life. So the choice of savings $b_{2,t+1}$ in the first period of live is determined by the intertemporal Euler equation in (2.11) and is represented below in its $S = 2$ version,

$$(c_{1,t})^{-\sigma} = \beta (1 + r_{t+1}) (c_{2,t+1})^{-\sigma} \quad \text{for } \forall t \quad (\text{T.2.2})$$

where the expectations operator is not present in (T.2.2) because all the components of $c_{1,t}$ are known at time t and all the components of the right-hand-side of are all endogenous or known.

Substituting the $s = 2$ solution from (T.2.1) into the right-hand-side of (T.2.2) and substituting the $s = 1$ constraint from (2.2) into the left-hand-side of (T.2.2) gives the following characterization of the optimal savings level $b_{2,t+1}$.

$$(w_t n_{1,t} - b_{2,t+1})^{-\sigma} = \beta (1 + r_{t+1}) [(1 + r_{t+1}) b_{2,t+1} + w_{t+1} n_{2,t+1}]^{-\sigma} \quad \text{for } \forall t \quad (\text{T.2.3})$$

Because the labor endowment is supplied inelastically $n_{s,t} = \bar{n}$, (??) implies that aggregate employment is the following.

$$N_t = \bar{n} \quad \forall t \quad (\text{T.2.4})$$

And because households only save during one period of their lives $b_{2,t+1}$, (??) implies that the aggregate capital level is the following.

$$K_t = \frac{1}{2} b_{2,t} \quad \forall t \quad (\text{T.2.5})$$

The production function (2.12) and the two first order conditions of the firm (2.13) and (2.14), combined with (T.2.4) and (T.2.5), show that the real wage w_t is simply a function of the savings by the previous generation in the previous period $b_{2,t}$. These equations also show that the rental rate on capital and the real wage in the next period r_{t+1} and w_{t+1} are simply functions of the endogenous savings rate of the current period households $b_{2,t+1}$.

$$w_t = (1 - \alpha) \left(\frac{b_{2,t}}{2\bar{n}} \right)^\alpha \quad (\text{T.2.6})$$

$$r_{t+1} = \alpha \left(\frac{2\bar{n}}{b_{2,t+1}} \right)^{1-\alpha} \quad (\text{T.2.7})$$

$$w_{t+1} = (1 - \alpha) \left(\frac{b_{2,t+1}}{2\bar{n}} \right)^\alpha \quad (\text{T.2.8})$$

So the optimal level of savings in each period $b_{2,t+1}$ can now be expressed as a function of the state, which is the savings from the previous period $b_{2,t}$, by solving the following version of the intertemporal Euler equation for $b_{2,t+1}$.

$$\beta \left[1 + \alpha \left(\frac{2\bar{n}}{b_{2,t+1}} \right)^{1-\alpha} \right] \left(\left[1 + \alpha \left(\frac{2\bar{n}}{b_{2,t+1}} \right)^{1-\alpha} \right] b_{2,t+1} + (1-\alpha) \left(\frac{b_{2,t+1}}{2} \right)^\alpha (\bar{n})^{1-\alpha} \right)^{-\sigma} = \dots \quad (\text{T.2.9})$$

Equation (T.2.9) has no general analytical solution and must be solved numerically, even in the log utility case in which $\sigma = 1$.

Figure 4: Savings Policy Function $b_{2,t+1}(b_{2,t})$: $S = 2$

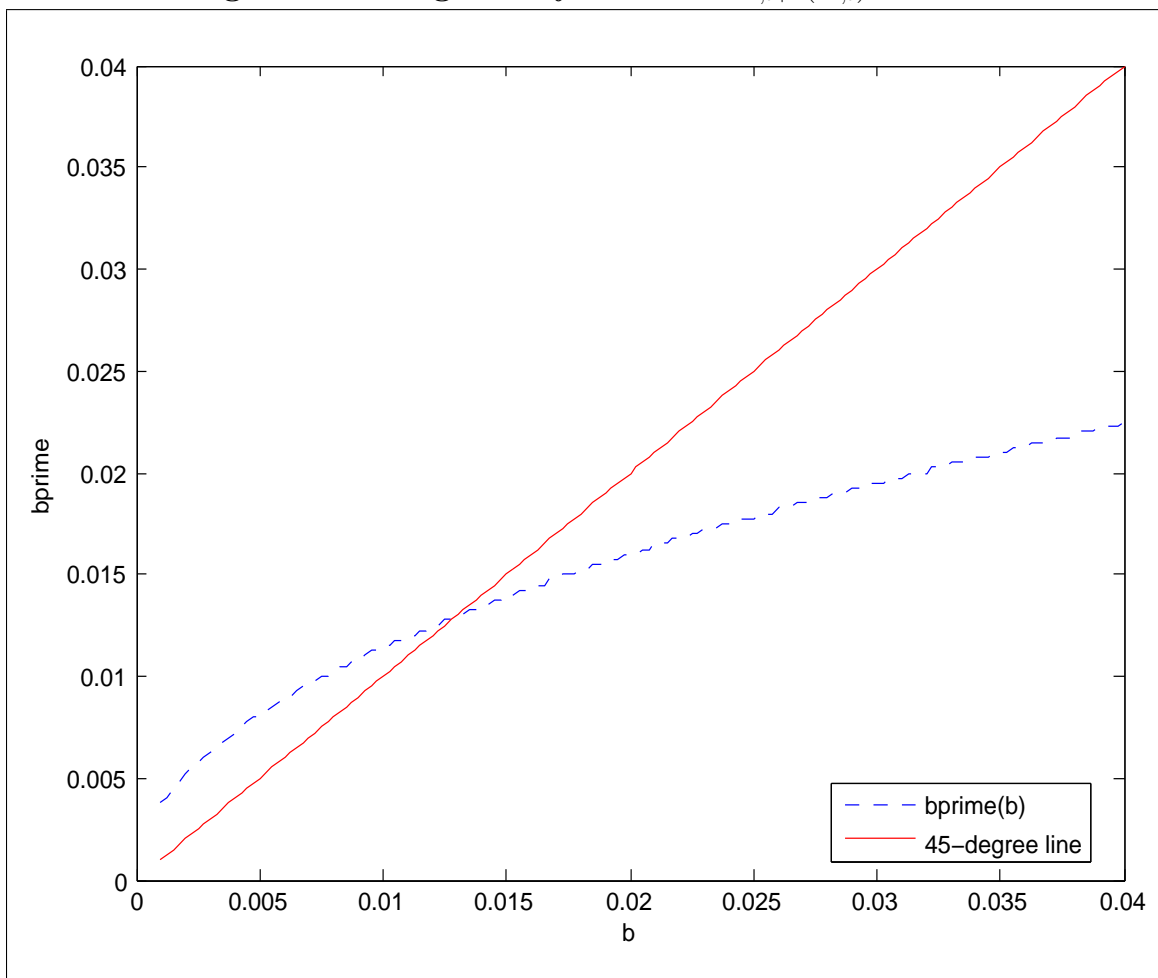


Figure 4 illustrates the policy function for $b_{2,t+1}$ as a function of the state $b_{2,t}$. The numerical solution was calculated setting $\beta = 0.6$, $\sigma = 2$, $A = 1$, $\alpha = 0.35$, $\bar{n} = 1$, and the state space $b_{2,t} \in [0.001, 0.04]$.⁴ The red line is the 45-degree line, and the

⁴The MatLab code for this numerical solution is available upon request.

point at which the policy function crosses the 45-degree line is the stable steady state of $\bar{b} = 0.013$.

The intuition of the policy function for individual savings $b_{2,t+1}$ is that the current period real wage w_t is an increasing function of last period's savings $b_{2,t}$. Thus, when last period's savings is high, the wage rate is high, and higher current period wages reduce the incentive to save. The opposite argument also holds that when last period's savings are low, wages are lower and the lower wages are compensated for by increased savings.

T-3 Example: Three-period lived agents $S = 3$

In the case of the $S = 3$ model, the aggregate capital and labor equations analogous to (??) and (??) take the following form.

$$N_t = \sum_{s=1}^3 \frac{1}{3} n_{s,t} = \bar{n} \quad \forall t \quad (\text{T.3.1})$$

$$K_t = \sum_{s=2}^3 \frac{1}{3} b_{s,t} \quad \forall t \quad (\text{T.3.2})$$

The production function (2.12) and the two first order conditions of the firm (2.13) and (2.14), combined with (T.3.1) and (T.3.2), show that the real wage w_t is simply a function of the savings by the previous two generations in the previous period $b_{2,t}$ and $b_{3,t}$. These equations also show that the rental rate on capital and the real wage in the next period r_{t+1} and w_{t+1} are simply functions of the endogenous savings rate of the current period new households ($s = 1$) and the current period middle aged households ($s = 2$) $b_{2,t+1}$ and $b_{3,t+1}$.

$$w_t = (1 - \alpha) \left(\frac{\sum_{s=2}^3 \frac{1}{3} b_{s,t}}{\bar{n}} \right)^\alpha \quad (\text{T.3.3})$$

$$r_t = \alpha \left(\frac{\bar{n}}{\sum_{s=2}^3 \frac{1}{3} b_{s,t}} \right)^{1-\alpha} \quad (\text{T.3.4})$$

$$w_{t+1} = (1 - \alpha) \left(\frac{\sum_{s=2}^3 \frac{1}{3} b_{s,t+1}}{\bar{n}} \right)^\alpha \quad (\text{T.3.5})$$

$$r_{t+1} = \alpha \left(\frac{\bar{n}}{\sum_{s=2}^3 \frac{1}{3} b_{s,t+1}} \right)^{1-\alpha} \quad (\text{T.3.6})$$

As in (2.9), consumption in the last period of life $c_{3,t}$ is the following expression,

$$c_{3,t} = (1 + r_t) b_{3,t} + w_t n_{3,t} \quad \forall t \quad (\text{T.3.7})$$

and is simply a function of the savings of that household in the previous period $b_{3,t}$ and the savings of the previous generation household in the previous period $b_{2,t}$.

$$c_{3,t} = \left[1 + \alpha \left(\frac{\bar{n}}{\sum_{s=2}^3 \frac{1}{3} b_{s,t}} \right)^{1-\alpha} \right] b_{3,t} + (1 - \alpha) \left(\sum_{s=2}^3 \frac{1}{3} b_{s,t} \right)^\alpha (\bar{n})^{1-\alpha} \quad \forall t \quad (\text{T.3.8})$$

There is no choice to be made in the last period of life. Households simply consume their income which depends only on savings decisions made in the previous period $b_{2,t}$ and $b_{3,t}$.

$$c_{3,t} = c_{3,t}(b_{2,t}, b_{3,t}) \quad \text{s.t.} \quad (\text{T.3.7}) \text{ holds} \quad (\text{T.3.9})$$

When $s = 2$, the two-period-old household takes as given the real rental rate r_t and the real wage rate w_t in that period and chooses how much to consume $c_{2,t}$ and how much to save for the next period $b_{3,t+1}$. The decision is governed by the intertemporal Euler equation (2.11) and takes the following form.

$$(c_{2,t})^{-\sigma} = \beta(1 + r_{t+1})(c_{3,t+1})^{-\sigma} \quad \forall t \quad (\text{T.3.10})$$

From (T.3.9), $c_{3,t+1}$ is simply a function of the two-period-old household's savings decision in the current period $b_{3,t+1}$ and the current-period savings decision of the one-period-old household $b_{2,t+1}$. Substituting the solution for consumption in the last period of life $c_{3,t+1}$ from (T.3.7) into the right-hand-side of (T.3.10) and substituting the per-period budget constraint from (2.3) into the left-hand-side of (T.3.10) gives the following expression.

$$\begin{aligned} & [(1 + r_t)b_{2,t} + w_t n_{2,t} - b_{3,t+1}]^{-\sigma} = \dots \\ & \beta(1 + r_{t+1}) [(1 + r_{t+1})b_{3,t+1} + w_{t+1} n_{3,t+1}]^{-\sigma} \quad \forall t \end{aligned} \quad (\text{T.3.11})$$

From (T.3.3) and (T.3.4), w_t and r_t on the left-hand-side of (T.3.11) are functions of the state, which consists of the two-period-old household's savings from the last period $b_{2,t}$ and the savings of the three-period-old household from the last period $b_{3,t}$. From (T.3.8), the right-hand-side of (T.3.11) is simply a function of the endogenous savings choice $b_{3,t+1}$ and beliefs about what the one-period-old generation will save for next period $b_{2,t+1}$. I assume a symmetric equilibrium among households in that all households of any generation s in any period t have the same savings policy function for that period of life.

$$b_{s,t+1}(\cdot) = b_{s,t}(\cdot) \quad \forall s, t \quad (\text{T.3.12})$$

Now, given that two-period-old households know the savings policy function of the one-period-old households $b_{2,t+1}(\cdot)$ as a function of the state, the choice of savings in the second period of life $b_{3,t+1}$ is simply a function of the state $b_{2,t}$ and $b_{3,t}$.

$$b_{3,t+1} = b_{3,t+1}(b_{2,t}, b_{3,t}) \quad \text{s.t.} \quad (\text{T.3.11}) \text{ holds} \quad (\text{T.3.13})$$

Using the per-period budget constraint (2.3) and the policy function for savings in the second period of life (T.3.13) the policy function for consumption in the second period of life $c_{2,t}$ can be written as a function of the state.

$$c_{2,t} = c_{2,t}(b_{2,t}, b_{3,t}) \quad \text{s.t.} \quad (\text{2.3}) \text{ and } (\text{T.3.7}) \text{ hold} \quad (\text{T.3.14})$$

The last step is the first-period-of-life savings decision $b_{2,t+1}$ which is governed by the intertemporal Euler equation (2.11) which takes the following form,

$$(c_{1,t})^{-\sigma} = \beta(1 + r_{t+1})(c_{2,t+1})^{-\sigma} \quad \forall t \quad (\text{T.3.15})$$

From (T.3.14), $c_{2,t+1}$ is simply a function of the two-period-old household's savings decision in the current period $b_{3,t+1}$ and the current-period savings decision of the one-period-old household $b_{2,t+1}$. In turn, (T.3.13) and the symmetric equilibrium

assumption in (T.3.12) show that $b_{2,t+1}$ and $b_{3,t+1}$ are also simply functions of the state $b_{2,t}$ and $b_{3,t}$. Substituting the policy function for consumption in the second period of life $c_{2,t+1}$ from (T.3.14) into the right-hand-side of (T.3.10) and substituting the per-period budget constraint from (2.2) into the left-hand-side of (T.3.10) gives the following expression.

$$(w_t n_{1,t} - b_{2,t+1})^{-\sigma} = \beta (1 + r_{t+1}) [c_{2,t+1} (b_{2,t+1}, b_{3,t+1})]^{-\sigma} \quad \forall t \quad (\text{T.3.16})$$

Everything on the right-hand-side of (T.3.16) is a function of the choice variable $b_{2,t+1}$ and two-period old household's savings decision $b_{3,t+1}$. The symmetric equilibrium assumption from (T.3.12) implies that if a household knows its own policy function for savings in the second period of life $b_{3,t+1} (b_{2,t}, b_{3,t})$, then it knows the policy function of other generation households for savings in the second period of their lives $b_{3,u+1} (b_{2,u}, b_{3,u})$ for all $u \neq t$. Thus, the one-period-old household knows that the two-period-old household's savings decision will be a known function of the state $b_{3,t+1} = b_{3,t+1} (b_{2,t}, b_{3,t})$. Everything on the left-hand-side of (T.3.16), apart from the choice variable $b_{2,t+1}$, is also a function of the state. Therefore, the policy function for $b_{2,t+1}$ can be written as a function of the state.

$$b_{2,t+1} = b_{2,t+1} (b_{2,t}, b_{3,t}) \quad \text{s.t.} \quad (\text{T.3.16}) \text{ holds} \quad (\text{T.3.17})$$

Using the per-period budget constraint (2.2) and the policy function for savings in the first period of life (T.3.17) the policy function for consumption in the first period of life $c_{1,t}$ can be written as a function of the state.

$$c_{1,t} = c_{1,t} (b_{2,t}, b_{3,t}) \quad \text{s.t.} \quad (2.2) \text{ and } (\text{T.3.16}) \text{ hold} \quad (\text{T.3.18})$$

T-3.1 Distribution of capital time path fixed point solution

The most common method of solving for the savings policy functions as functions of the state $b_{2,t} (b_{t-1})$ and $b_{3,t} (b_{t-1})$ is by finding a fixed point for the time-path of the distribution of capital for each possible initial state of the distribution of capital.

The first step is to assume a time path for aggregate capital $K^0 = \{K_1^0, K_2^0, \dots, K_T^0\}$. This time path for aggregate capital determines the time path for both the real wage $w^0 = \{w_1^0, w_2^0, \dots, w_T^0\}$ and the real return on investment $r^0 = \{r_1^0, r_2^0, \dots, r_T^0\}$. The exact initial distribution of capital in the first period $b_1 = \{b_{2,1}, b_{3,1}\}$ such that $K_1^0 = \sum_{s=2}^3 \frac{1}{3} b_{s,1}$ according to (T.3.2). One could also first choose the initial distribution of capital $b_1 = \{b_{2,1}, b_{3,1}\}$ and then choose an initial aggregate capital stock K_1^0 that corresponds to that distribution. But the aggregate capital stocks for periods $t > 1$ can be any level.

Given the initial capital distribution $b_1 = \{b_{2,1}, b_{3,1}\}$ and the time paths of aggregate capital $K^0 = \{K_1^0, K_2^0, \dots, K_T^0\}$, the real wage $w^0 = \{w_1^0, w_2^0, \dots, w_T^0\}$, and the real return to investment $r^0 = \{r_1^0, r_2^0, \dots, r_T^0\}$, one can solve for the optimal level of savings by the two-period-old agent for the third period of his life $b_{3,2}$ using his intertemporal Euler equation:

$$[(1 + r_1^0) b_{2,1} + w_1^0 n_{2,1} - b_{3,2}]^{-\sigma} = \beta (1 + r_2^0) [(1 + r_2^0) b_{3,2} + w_2^0 n_{3,2}]^{-\sigma} \quad (\text{T.3.19})$$

Because r_1^0 , w_1^0 and $b_{2,1}$ were decided by the choice of initial state and because r_2^0 and w_2^0 were assumed, $b_{3,2}$ is the savings value that makes (T.3.19) hold.

The values for the one-period-old agent's savings in period 1 for the next period of life $b_{2,2}$ as well as his savings in period 2 for the last period of life $b_{3,3}$ are solved for jointly using his two intertemporal Euler equations:

$$[w_1^0 n_{1,1} - b_{2,2}]^{-\sigma} = \beta (1 + r_2^0) [(1 + r_2^0) b_{2,2} + w_2^0 n_{2,2} - b_{3,3}]^{-\sigma} \quad (\text{T.3.20})$$

$$[(1 + r_2^0) b_{2,2} + w_2^0 n_{2,2} - b_{3,3}]^{-\sigma} = \beta (1 + r_3^0) [(1 + r_3^0) b_{3,3} + w_3^0 n_{3,3}]^{-\sigma} \quad (\text{T.3.21})$$

Because w_1^0 was decided by the choice of initial state and because r_2^0 , r_3^0 , w_2^0 , and w_3^0 were assumed, $b_{2,2}$ and $b_{3,3}$ are the levels of savings that make both (T.3.20) and (T.3.21) hold.

With optimal solutions for $b_{2,2}$ and $b_{3,2}$ given the assumed time paths of capital K^0 , the real wage w^0 , the return on investment r^0 , and the initial distribution of capital $b_1 = \{b_{2,1}, b_{3,1}\}$, a new period-2 level of aggregate capital $K_2^{0'} = \sum_{s=2}^3 \frac{1}{3} b_{s,2}$ can be calculated. This will also give a new period-2 real wage $w_2^{0'}$ and real return on investment $r_2^{0'}$. In general, the value of the new calculated period-2 level of aggregate capital is not equal to the assumed period-2 level of capital $K_2^{0'} \neq K_2^0$.

Iterating one period forward, $b_{2,3}$ and $b_{3,4}$ are found using the one-period-ahead versions of (T.3.20) and (T.3.21) given the new $K_2^{0'}$, $w_2^{0'}$, and $r_2^{0'}$ and the rest of the originally assumed time path of K^0 . In like manner, the optimally chosen $b_{2,3}$ and $b_{3,3}$ give a new period-3 aggregate capital level $K_3^{0'}$. This process is repeated until new values for the entire time path of aggregate capital are calculated $K^{0'} = \{K_1^0, K_2^{0'}, K_3^{0'}, \dots, K_T^{0'}\}$.

Once the new time path of aggregate capital $K^{0'}$ is calculated, a new time path is assumed that is a convex combination (a contraction) of the initially assumed time path K^0 and the newly calculated time path $K^{0'}$.

$$K^1 = \gamma K^{0'} + (1 - \gamma) K^0 \quad \forall \gamma \in (0, 1) \quad (\text{T.3.22})$$

The process of calculating new values for the time path of aggregate capital $K^{1'}$ is repeated, and the convex combination of the initial time path K^1 and the newly calculated time path $K^{1'}$ form a new time path of capital K^2 . For any positive integer s , the next guess for a time path of aggregate capital K^{s+1} is given by the following equation.

$$K^{s+1} = \gamma K^{s'} + (1 - \gamma) K^s \quad \forall s \geq 1, \gamma \in (0, 1) \quad (\text{T.3.23})$$

We define the equilibrium time path of the distribution of capital b_t^* as the distribution of capital that is consistent with the time path of the level of aggregate capital K^s . This equilibrium distribution of capital is found when the calculated time path of aggregate capital $K^{s'}$ is very close to the assumed time path of aggregate capital K^s .

Definition 6 (Equilibrium time path of the distribution of capital). Given some initial distribution of capital $b_1 = \{b_{2,1}, b_{3,1}\}$, the equilibrium time path of the distribution of capital $b^* (b_{2,1}, b_{3,1})$ is defined as the distribution of capital for which

the calculated level of aggregate capital $K^{s'}$ is arbitrarily close to the initial guess for the time path of aggregate capital K^s .

$$b^*(b_{2,1}, b_{3,1}) \equiv b(b_{2,1}, b_{3,1}) : |K^{s'} - K^s| < \varepsilon$$

where ε is a number arbitrarily close to zero.

T-3.2 Backward induction solution from steady state

Another method for solving this system is by using backward induction from the steady state distribution of capital $\bar{b} = \{\bar{b}_2, \bar{b}_3\}$. The steady state capital distribution \bar{b} is defined as follows.

Definition 7 (Definition of steady state distribution of capital in 3-period-lived agents model). The steady state distribution of capital \bar{b} is a distribution of capital $b_t = \{b_{2,t}, b_{3,t}\}$ such that $b_{t+1} = b_t$. This implies that in the steady state distribution of capital, $b_{s,t+1} = b_{s,t}$ for all s and t .

$$\bar{b} \equiv b_t : b_{t+1} = b_t \quad \forall t \quad \Rightarrow \quad b_{s,t+1} = b_{s,t} \quad \forall s, t$$

Using Definition 7 and (T.3.13) and (T.3.17) gives two equations and two unknowns that characterize the steady state savings rates \bar{b}_2 and \bar{b}_3 .

$$\begin{aligned} [\bar{w}n_1 - \bar{b}_2]^{-\sigma} &= \beta(1 + \bar{r}) [(1 + \bar{r})\bar{b}_2 + \bar{w}n_2 - \bar{b}_3]^{-\sigma} \\ [(1 + \bar{r})\bar{b}_2 + \bar{w}n_2 - \bar{b}_3]^{-\sigma} &= \beta(1 + \bar{r}) [(1 + \bar{r})\bar{b}_3 + \bar{w}n_3]^{-\sigma} \end{aligned} \quad (\text{T.3.24})$$

Notice that the real wage and the real return on investment have steady state values, \bar{w} and \bar{r} respectively, because (T.3.3) and (T.3.4) show that they are both simply functions of the distribution of capital b_t . The following proposition states that there exists a unique steady state capital distribution \bar{b} such that (T.3.24) holds.

Proposition 1 (Existence and Uniqueness of Steady State Distribution of Capital). Given the structure of the model characterized in equations (T.3.1) through (T.3.24), there exists a unique steady state distribution of capital $\bar{b} = \{\bar{b}_2, \bar{b}_3\}$ such that the system of $S - 1$ equations in (T.3.24) holds.

Proof. See Appendix A-1. □

With steady state savings decisions \bar{b}_2 and \bar{b}_3 (and therefore, steady state aggregate capital \bar{K}) characterized in Proposition 1, the entire policy functions for $b_{2,t+1}$ and $b_{3,t+2}$ are found by iterating backward, period by period, from the steady state. Assume that the economy reaches the steady state in period $t = T$. Then $b_{2,t} = \bar{b}_2$, $b_{3,t} = \bar{b}_3$ as described in Proposition 1, and $K_t = \bar{K}$ for all $t \geq T$. Let the savings decisions in period $T - 2$ be small deviations from the steady state savings decisions $b_{2,T-2} = \bar{b}_2 + \varepsilon_2$ and $b_{3,T-2} = \bar{b}_3 + \varepsilon_3$.

The savings decisions of the one-period-old and the two-period-old agents in period $T - 1$, respectively, are given by the following two intertemporal Euler equations in which the state of the distribution of capital is given by $b_{T-2} = \bar{b} + \varepsilon$.

$$[w_{T-2}n_{1,T-2} - b_{2,T-1}]^{-\sigma} = \beta (1 + r_{T-1}) [(1 + r_{T-1}) b_{2,T-1} + w_{T-1}n_{2,T-1} - \bar{b}_3]^{-\sigma} \quad (\text{T.3.25})$$

$$[(1 + r_{T-2}) b_{2,T-2} + w_{T-2}n_{2,T-2} - b_{3,T-1}]^{-\sigma} = \beta (1 + r_{T-1}) [(1 + r_{T-1}) b_{3,T-1} + w_{T-1}n_{3,T-1}]^{-\sigma} \quad (\text{T.3.26})$$

Given the state in period $T - 2$ of the distribution of capital $b_{T-2} = \bar{b} + \varepsilon$ and that the economy reaches the steady state distribution of capital \bar{b} in period T , the problem of the one-period-old agent in period $T - 2$ is to choose $b_{2,T-1}$ to maximize (T.3.25) for any given savings choice of the two-period-old agent $b_{3,T-1}$. And the two-period-old agent's problem is to choose $b_{3,T-1}$ to maximize (T.3.26), given the state \bar{b} and for any given savings choice of the one-period-old agent $b_{2,T-1}$. This dependence of the optimal savings choice of both agents on the choice of the other agents gives rise to the need for a definition of a per period Nash equilibrium.

Definition 8 (Backward induction per-period Nash equilibrium in 3-period-lived agents model). Let b_{t-1} be the known state of the distribution of capital in period $t - 1$, and let b_{t+1} be the known state of the distribution of capital in period $t + 1$. Let $b_{2,t}(b_{3,t})$ be the choice of $b_{2,t}$ that maximizes the intertemporal Euler equation,

$$[w_{t-1}n_{1,t-1} - b_{2,t}]^{-\sigma} = \beta (1 + r_t) [(1 + r_t) b_{2,t} + w_t n_{2,t} - b_{3,t+1}]^{-\sigma}$$

given the two-period-old agent's choice of savings $b_{3,t}$. And let $b_{3,t}(b_{2,t})$ be the choice of $b_{3,t}$ that maximizes the intertemporal Euler equation,

$$[(1 + r_{t-1}) b_{2,t-1} + w_{t-1}n_{2,t-1} - b_{3,t}]^{-\sigma} = \beta (1 + r_t) [(1 + r_t) b_{3,t} + w_t n_{3,t}]^{-\sigma}$$

given the one-period-old agent's choice of savings $b_{2,t}$. Then a per-period Nash equilibrium to the backward induction solution in the 3-period-lived agents model is the intersection of the best response functions $b_{2,t}(b_{3,t}) = b_{3,t}(b_{2,t})$.

The entire policy functions for $b_{2,t}(b_{t-1})$ and $b_{3,t}(b_{t-1})$ as functions of the state b_{t-1} are found by solving for the per-period Nash equilibrium solutions for period $T - 1$ savings given period $T - 2$ capital distributions in the ε -neighborhood of the steady state $b_{T-2} = \bar{b} + \varepsilon$. Once the per-period Nash equilibrium solution is found for the distribution of capital in period $T - 1$, the period $T - 2$ distribution of capital can be found by iterating the problem backward in time one period, letting the state of the distribution of capital in period $T - 3$ be an epsilon ball around the state used previously $b_{3,T-3} = \bar{b} + \varepsilon + \varepsilon$, and solving for the per-period Nash equilibrium $T - 2$ distribution of capital given the known solution for period $T - 1$. The policy function is solved by backward induction over the entire $S - 1$ dimensional state.⁵

⁵Note that this solution method is based on the assumption that all savings rates converge to the steady state, regardless of the state, i.e., $|b_{s,t} - \bar{b}_s| \geq |b_{s,t+1} - \bar{b}_s|$. We should prove this.

T-4 Comments and Notes

- John Rust has some social security, life-cycle model, computational papers on his web site (<http://gemini.econ.umd.edu/jrust/papers.html>). He was also a member of the Social Security Board's 1999 Technical Panel on Assumptions and Methods. His work would be a good reference.
- We could tie this stuff into the [Krusell and Smith \(1998\)](#) that the average wealth is a sufficient statistic for the entire distribution of wealth for making decisions about household future investment. We could see how close our forecasting method gets to the benchmark TPI method when the mean savings amount is used as the forecast rather than some more complicated function of the entire distribution.
- [Gustman and Steinmeier \(2008\)](#) try to estimate the effects of a change in Social Security rules in the early 1990s. We should see if we can replicate their results with this model.